Tutorial *Analysis I*

Return: 12:00, Friday 21.12.

Exercise 25:

The series $\sum a_n$ is absolutely convergent if and only if its terms can be written in the form $a_n = b_n - c_n$, where $b_n, c_n \geq 0$ for all n and the series $\sum b_n$ and $\sum c_n$ converge.

Exercise 26:

Show: The series $\sum a_n$ is absolutely convergent if and only if for every bounded sequence (b_n) the series $\sum a_n b_n$ converges.

Exercise 27:

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers with $|a_n| \leq M$ for all n. Show:

(i) For every $x \in \mathbb{R}$ with $|x| < 1$, the following series is convergent:

$$f(x) = \sum_{n=1}^{\infty} a_n x^n.$$

(ii) If $a_1 \neq 0$, then $f(x) \neq 0$ for all $x \in \mathbb{R}$ with $0 < |x| < \frac{|a_1|}{2M}$.

Exercise 28:

The exponential function is defined by $\exp(x) = \sum_{n=0}^{\infty} x^n/n!$. Show that for every $x \in \mathbb{R}$

$$\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n.$$

Exercise 29:

The Riemann Zeta function

$$\zeta(r) = \sum_{n=1}^{\infty} \frac{1}{n^r}, \qquad r > 1$$

should be approximated up to a given tolerance $\epsilon > 0$ by using partial sums

$$\zeta_\epsilon(r) = \sum_{n=1}^{N(\epsilon,r)} \frac{1}{n^r}, \qquad r > 1.$$

Determine a suitable function $(\epsilon, r) \mapsto N(\epsilon, r) \in \mathbb{N}$, such that $|\zeta(r) - \zeta_\epsilon(r)| < \epsilon$ for all $r > 1$. Generate a graphical representation of $\zeta_{0.1}$ on the interval $1.3 \leq r \leq 5$ (using a small computer program).