UNIVERSITÄT KAISERSLAUTERN

Fachbereich Mathematik

Distributions & Fourier transform

WS 1999/2000

Contents

Chapter 0

Introduction

The first question we have to ask is: What are distributions?

Distributions are generalized functions! But in which way are distributions generalized? And why is such a generalization necessary? Therefor let us first condsider some examples:

0.1 Some guitar playing

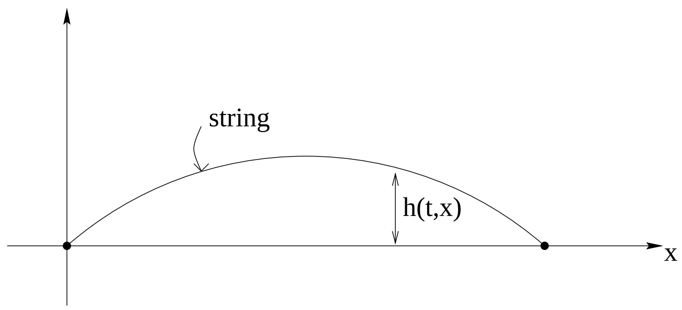


Figure 1: guitar string

$h(t,x)$ = deviation from rest position depending on time t, position x, governed by:

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2} \qquad \text{(wave equation)}$$

$$+ \text{ initial \& boundary conditions}$$

Real picking the string means:

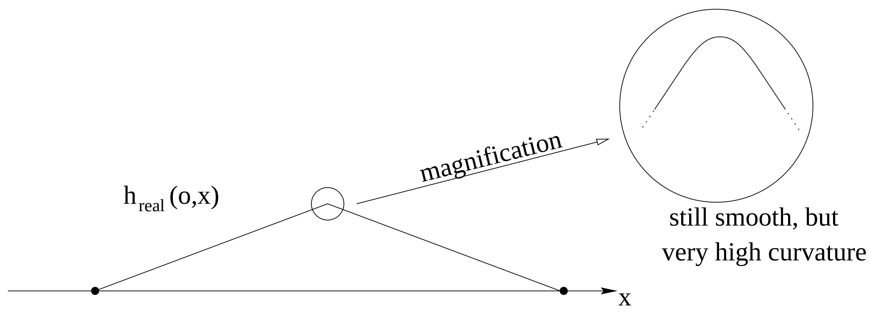


Figure 2: real picking

Ideal picking: To avoid the complicated description of the initial value h_{real},

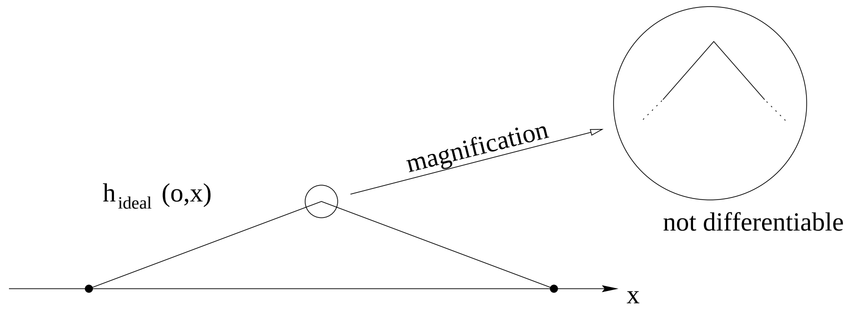


Figure 3: ideal picking

one wants to use the simpler condition h_{ideal}.

Disadvantage: How can the wave equation be satisfied at $t = 0$:

$$\left.\frac{\partial^2 h}{\partial x^2}\right|_{t=0} = ??$$

0.2 An ecological crime

We consider the temperature distribution in a large room with central heating and aircondition. Figure 4 shows the idealized heat production (positive part = heating, negative part = air conditioning). This problem is governed by:

$$0 = \lambda \frac{\partial^2 T}{\partial x^2} + q_{\text{ideal}} \qquad \text{stationary heat equation}$$

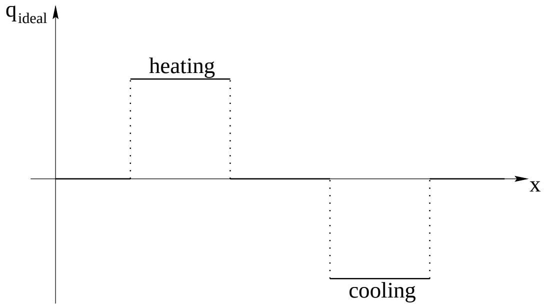


Figure 4: Heat production q_{ideal}

λ:	heat conduction coefficient
T:	temperature
q_{ideal}:	heat production

Disadvantage: How can $\frac{\partial^2 T}{\partial x^2}$ have jumps?
This problem could be avoided by replacing q_{ideal} by a "smeared out" approximation:

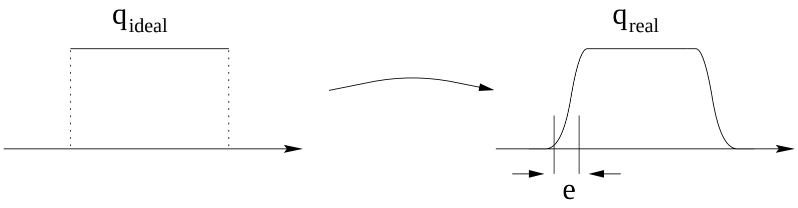


Figure 5: smoothing jumps

But this is again more difficult to describe, as we have an extra parameter ε which is (probably) not relevant for the solution!

In 0.1 and 0.2 we have differential equations with non-smooth data (because of idealizations, simplifications $\rightarrow$ laziness). This leads to problems with the classical definition of differentiability.

Conclusions: a) There is no free lunch
b) New concept of differentiability required

Many further examples available ...

0.3 Motion of an asteroid

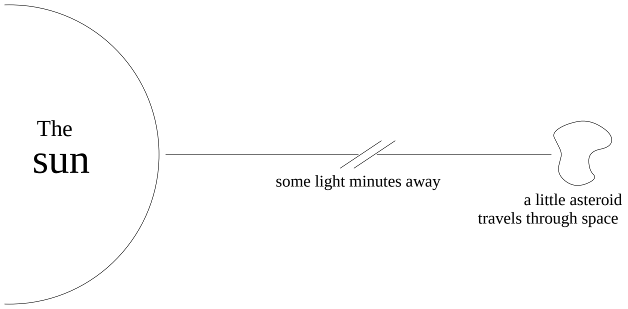


Figure 6: Setup of example 3

The problem is now to describe the motion of the asteroid. In problem-relevant scales ($\sim$ Mio km), the shape of the asteroid is "irrelevant". This

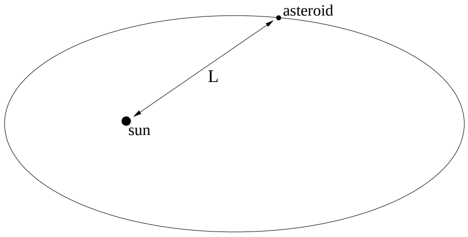


Figure 7: Radius of the asteroid can be neglected

can be seen by calculating the total force acting on the asteroid (assuming the sun is spherical and uniform with center in $x = 0$ and total mass M):

$$F(x) = -GM \int\limits_A \frac{\rho(y)}{|x+y|^3}(x+y)\mathrm{d}y$$

Now go to the correct scale:

$$x = L\bar{x}, \quad y = Lz, \quad \mathrm{d}y = L^3\mathrm{d}z$$

$$
\begin{aligned}
F(x) &= -\frac{GM}{L^2}\int\limits_{A/L} \frac{L^3\rho(Lz)}{|\bar{x}+z|^3}(\bar{x}+z)\mathrm{d}z \\
&\overset{z\text{ very small}}{\approx} -\frac{GM}{L^2}\frac{\bar{x}}{|\bar{x}|^3}\underbrace{\int\limits_{A/L} L^3\rho(Lz)\mathrm{d}z}_{m} \\
&= -\frac{GMm}{L^2}\frac{\bar{x}}{|\bar{x}|^3}
\end{aligned}
$$

where m is the total mass of the asteroid. Then we can solve $\ddot{\bar{x}} = F$ to get

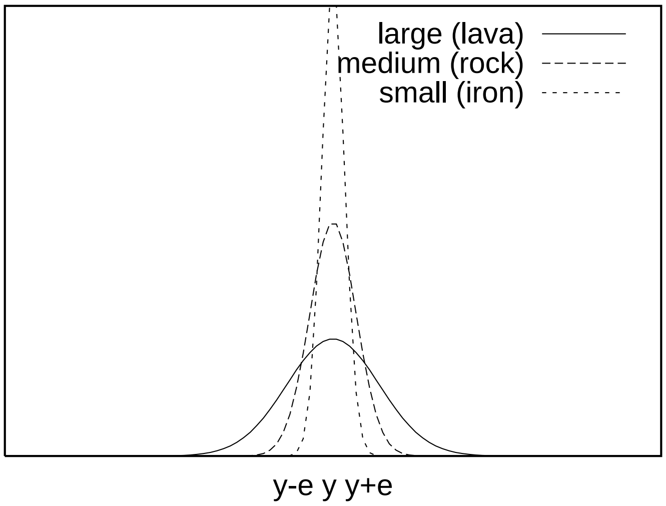


Figure 8: Different asteroid radii

the motion.

Here we use that for big L, the asteroid can be approximated by a mass-point. Mathematically, this can be written as:

$$L^3\rho(Lz) \overset{L\to\infty}{\longrightarrow} m\text{“}\delta\text{”}$$

such that

$$\int\limits_{\mathbb{R}^3} \delta(z)\mathrm{d}z = 1$$

In which sense does this converge?

0.4 Strategies for generalization

Thus we come to the central question. We have seen in some examples that if we simplify problems, we sometimes cant solve them with our classical mathematical means. So what should be generalized? We have seen that we need at least generalized differentiation, but more general, we want to generalize classical calculus for functions with several variables, in particular *differentiation, integration, convolution* and *Fourier transform.*

First we are going to present an example how we should not generalize:

0.4.1 Bad generalization of derivatives

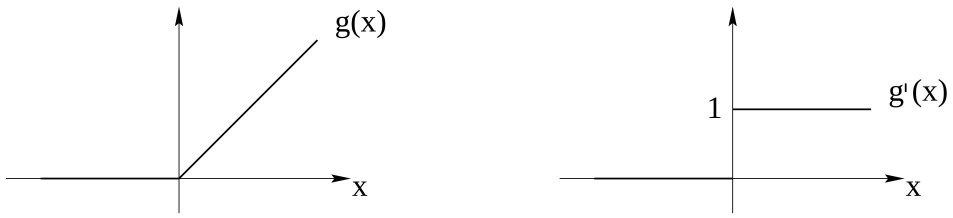


Figure 9: $g(x) = x^+$ and Heaviside function

Consider the function $g(x) = x^+$ (see figure 9). Obviously with classical mathematics, g is not differentiable in $x = 0$. To solve this problem, one could simply “omit this point:

“Natural” first derivative:

$$g'(x) = H(x) = \begin{cases} 1 & : \quad x \geq 0 \\ 0 & : \quad x < 0 \end{cases}$$

H is called *Heaviside function* (see figure 9).

“Natural” second derivative:

$$g''(x) = 0$$

Here we see immediately the huge disadvantage of this proceeding: The fundamental theorem of calculus (FTC) is no longer valid:

$$0 = \int_{-1}^{1} g''(x)\mathrm{d}x \neq g'(1) - g'(-1) = 1$$

Remark
FTC cannot be recovered if H' is a classical function, as we need:

$$\int_a^b H'(x)\mathrm{d}x = \begin{cases} 1 & : \quad 0 \in (a,b) \\ 0 & : \quad otherwise \end{cases}$$

This means that H has to be zero everywhere outside of 0. How can we accomplish this while satisfying FTC?

0.4.2 A good generalization (of derivatives)

When do we know a function $g : \mathbb{R} \to \mathbb{R}$? The answer is of course: If we know all values $g(x)$, $x \in \mathbb{R}$. But we also know it if we know all local averages of g, i.e.

$$\frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} g(\bar{x} + x)\mathrm{d}x = \int g(x)\Psi_\varepsilon(x)\mathrm{d}x$$

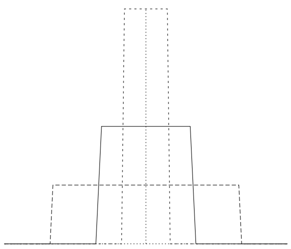
where

$$\Psi_\varepsilon(x) = \frac{1}{\varepsilon}\mathbb{1}_{(-\varepsilon/2,\varepsilon/2)}(x - \bar{x})$$

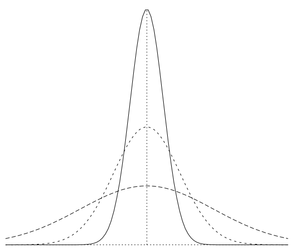
Observe:

$$g \text{ continuous } \Rightarrow \quad \lim_{\varepsilon \to 0} \int g(x)\Psi_\varepsilon(x)\mathrm{d}x = g(\bar{x})$$

Instead of $\Psi_\varepsilon =$



one can also take $\Psi_\varepsilon =$



i.e. functions which are smooth with compact support and $\int \Psi_\varepsilon \mathrm{d}x = 1$

First Important Observation

$$\boxed{g \widehat{=} T_g}$$

i.e. We know a function $g \in \mathbb{L}^1_{loc}(\mathbb{R})$ if we know all integral values

$$T_g(\Psi) = \int g(x)\Psi(x)\mathrm{d}x$$

for $\Psi \in C_0^\infty(\mathbb{R})$ (infinitely often differentiable with compact support).

Let $g \in C^1(\mathbb{R})$, then

$$\begin{aligned}
T_{g'}(\Psi) = \int g'(x)\Psi(x)\mathrm{d}x \quad & \overset{\Psi\text{smooth}}{=} \quad \int \left((g\Psi)' - g\Psi'\right)\mathrm{d}x \\
& \underset{\text{support}}{\overset{\Psi\text{compact}}{=}} \quad \int_{-R}^{R} \left((g\Psi)' - g\Psi'\right)\mathrm{d}x \qquad \text{for } R \text{ sufficiently large} \\
\ldots \quad & \overset{\text{FTC}}{=} \quad (g\Psi)'\big|_{-R}^{R} - \int_{-R}^{R} g\Psi'\mathrm{d}x \\
& = \quad -T_g(\Psi')
\end{aligned}$$

The derivative $g' \widehat{=} T_{g'}$ is given by

$$(T_g)'(\Psi) := -T_g(\Psi')$$

Second Important Observation

$$\boxed{g' \widehat{=} (T_g)'}$$

We can identify the derivative of g with $g' \widehat{=} (T_g)'$. In particular this makes sense even for nonsmooth g!

Now we want to apply this concept to our former example $g(x) = x^+$:

$$(T_g)'(\psi) = -T_g(\psi') = \int_0^\infty x\psi'(x)\mathrm{d}z = \int_0^\infty \psi(x)\mathrm{d}x = \int_0^\infty H(x)\psi(x)\mathrm{d}x = T_H(\psi)$$

Remark
That means $(T_g)'(\psi) = T_H(\psi),\ \forall\psi \in C_0^\infty(\mathbb{R})$. *Hence,* $(T_g)' = T_H$*, or in a more sloppy notation* $g' = H$*.*

For the derivative of the Heaviside function H we have

$$(T_H)'(\psi) = -T_H(\psi') = -\int_0^\infty \psi'(x)\mathrm{d}x \overset{\text{FTC}}{=} \psi(0)$$

If we now define $\delta(\psi) := \psi(0)$ then we get $(T_H)' = \delta$.

Note: $\delta \neq 0$ (as it was the case in the bad generalization).

Remark
A very important fact is that δ *cannot be identified with a classical function as in the previous examples, i.e. there exists no* $h \in \mathbb{L}^1_{loc}$ *such that* $\delta = T_h$ *.*

So the main point in theses considerations is that we want to use a concept of *generalized functions*, which are *functionals* which operate on $\psi \in C_0^\infty$.

We will see that this concept turns out to work!

0.5 A Scanner

In this last example of the introduction we want to show that these generalized functions are very natural objects when talking about physical experiments.

Consider the real example of a scanner:

The light sensitive device is in our case a photo-cell as in figure 11 where one can see a cell with a high resolution and one with a low resolution. The characteristic of the photo-cell is described by a function ψ (figure 12). The resolution of the cell depends on the support of ψ which is in general a radial symmetric function on x and y.

We want to measure the intensity $I(x, y)$ on the sensitive device but we can only measure

$$\iint I(x,y)\psi(x,y)\mathrm{d}x\mathrm{d}y \quad \text{with} \quad \iint \psi \mathrm{d}x\mathrm{d}y = 1$$

This is very typical for physical experiments. Only local averages can be determined.

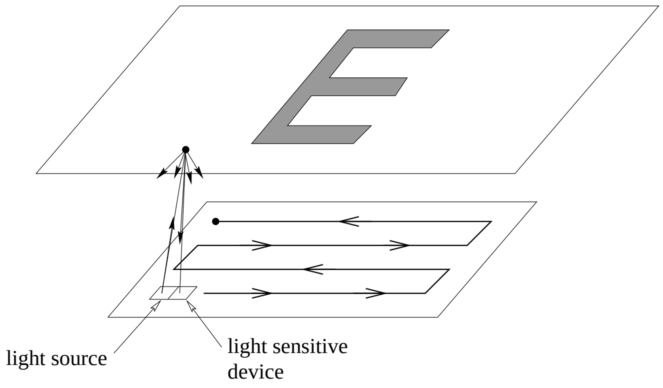


Figure 10: Sketch of a scanner

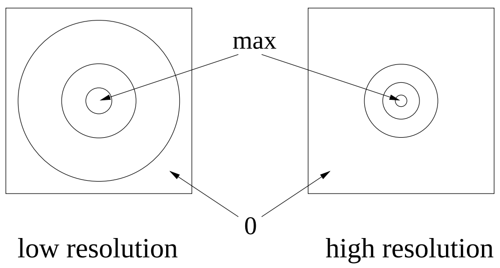


Figure 11: Low and high resolution

Note: local average = functional value $T_I(\psi)$ of the physical quantity I.

Terminology: In the following we say the intensity is "tested" at different positions. The scanning yields $T_I(\psi_n)$, $n = 1, 2, \ldots, N$. The functions ψ_n are called "test functions".

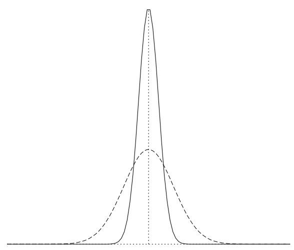


Figure 12: Ψ for low and high resolution

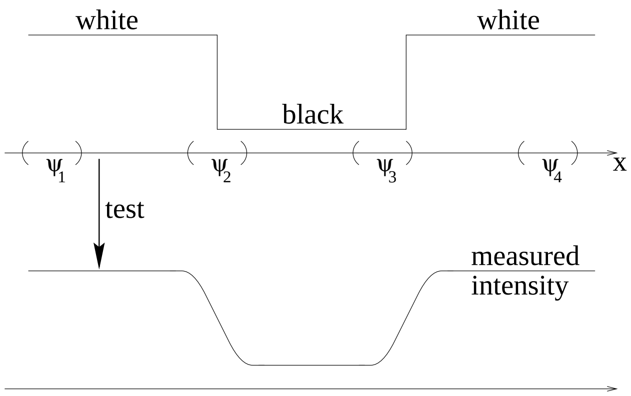


Figure 13: Real and measured intensities

Chapter 1

Test Functions

1.1 Function Spaces

Definition 1
Assume, $\varphi : \Omega \to C$. Then support of function φ is a set $supp\,\varphi \subset \Omega$:

$$supp\,\varphi := cl\{x \in \Omega | \varphi(x) \neq 0\}$$

Example
Take particular case $\Omega \subset \mathbb{R}^1$, $\Omega = (-20, 20)$.

$$\varphi(x) = \begin{cases} 0 & , \quad x \leq -\pi \\ \sin x & , \quad -\pi < x \leq \pi \\ 0 & , \quad \pi < x \leq 10 \\ x - 10 & , \quad 10 < x \leq 11 \\ -x + 12 & , \quad 11 < x \leq 12 \\ 0 & , \quad 12 < x \leq 13 \\ (x-13)^2 & , \quad x > 13 \end{cases}$$

Then $supp\,\varphi = [-\pi, \pi] \cup [10, 12] \cup [13; 20)$. *(See figure 1.1 in page 17.)*

Notation: We will write that $\varphi \in C_0^k(\Omega)$, if $\varphi : \Omega \to \mathbb{C}$ s.t.

1. $\varphi \in C^k(\Omega)$
2. $supp\,\varphi$ is compact

Definition 2
$\varphi : \Omega \to \mathbb{C}$ is a test function, if $\varphi \in C_0^\infty(\Omega)$.

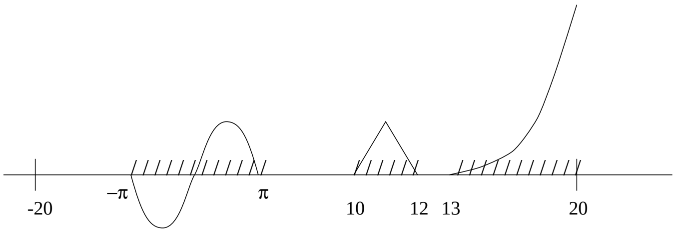


Figure 1.1: Support of function φ

Notation: $D(\Omega)$ is a set of all test functions:

$$D(\Omega) := C_0^\infty(\Omega)$$

Lemma 1
$D(\Omega)$ is a subspace of the vector space of complex valued functions on Ω.

Proof:

$C^\infty(\Omega)$ is a vector space. Since $D(\Omega) := C_0^\infty(\Omega) \subset C^\infty(\Omega)$, we have to check only 3 properties of vector spaces:

1. We must check if $D(\Omega) \neq \emptyset$. It is obviously, because $0 \in D(\Omega)$.
2. Check if for all $\varphi \in D(\Omega)$ and λ - scalar

   $$\lambda\varphi \in D(\Omega)$$

   It is true, because

   $$\varphi \in C^\infty(\Omega) \Rightarrow \lambda\varphi \in C^\infty(\Omega)$$

   and

   $$\begin{aligned} supp\,\lambda\varphi &= cl\{x \in \Omega | \lambda\varphi(x) \neq 0\} \\ &= cl\{x \in \Omega | \varphi(x) \neq 0, \text{ if } \lambda \neq 0\} = supp\,\varphi \end{aligned}$$

   if $\lambda \neq 0$. If $\lambda = 0$, then from 1-st point follows, that $\lambda\varphi = 0 \in D(\Omega)$.
3. Check, if for all $\varphi, \psi \in D(\Omega)$, $\varphi + \psi \in D(\Omega)$. It is true, because $\varphi + \psi \in C^\infty(\Omega)$ and

   $$\begin{aligned} supp(\varphi + \psi) &= cl\{x \in \Omega | (\varphi + \psi)(x) \neq 0\} \\ &\in cl\{x \in \Omega | \varphi(x) \neq 0 \text{ or } \psi(x) \neq 0\} \\ &= supp\,\varphi \cup supp\,\psi \end{aligned}$$

   Finite union of compact sets is compact, and $supp(\varphi + \psi)$ is closed set, included in compact set, from this follows, that $supp(\varphi + \psi)$ is compact set.

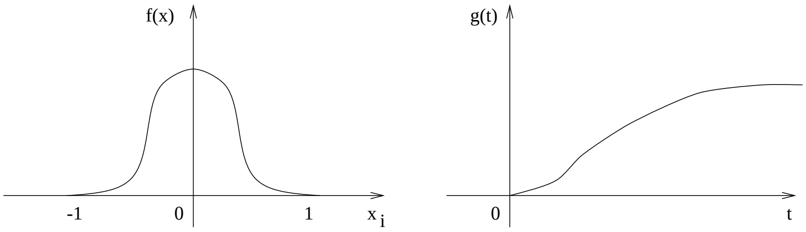


Figure 1.2: Functions f and g in remark

Remark
It would not be worth to consider $D(\Omega)$, if it consists of only one element. Therefore let us look for more interesting functions. Consider function

$$f(x) := \begin{cases} \exp\left(-\dfrac{1}{1-|x|^2}\right) & , \textit{if } |x| < 1 \\ 0 & , \textit{else} \end{cases} \quad \textit{where } x \in \mathbb{R}^d,$$

with

$$|x| = \left(\sum_i^d x_i^2\right)^{1/2} \quad \textit{– Euclidean norm}$$

We must check, that $f \in C^\infty(\mathbb{R}^d)$.

Note, that $f(x) = g(1-|x|^2)$, where

$$g(t) = \begin{cases} \exp(-1/t) & , \textit{if } t > 0 \\ 0 & , \textit{else} \end{cases}$$

$g(t) \in C^\infty(R)$, because $\forall n \in N$ exists $\left(\dfrac{d}{dt}\right)^n g\Big|_{t=0}$.

Generally, $f(x)$ can be outside from our $D(\Omega)$. To find "interesting" function in $D(\Omega)$, take $\bar{x} \in \Omega$. Then there exists $\bar{\varepsilon} > 0 : B_{\bar{\varepsilon}}(\bar{x}) \subset \Omega$. Here $B_{\bar{\varepsilon}}(\bar{x})$ is ball of radius $\bar{\varepsilon}$ arround $\bar{x}$. Define new function

$$\hat{f}(x) := f\left(\frac{x-\bar{x}}{\varepsilon}\right), \ \varepsilon < \bar{\varepsilon}\ , \text{ e.g., take } \varepsilon = \frac{\bar{\varepsilon}}{2}$$

Hence $\hat{f} \in D(\Omega)$.

1.2 Definition of Topology on $D(\Omega)$

$D(\Omega)$ is a linear space (test functions can be added and multiplied by real numbers to yield new test functions).

1.2.1 Definition of convergent sequences

Definition 3
We shall say that a sequence $\varphi_1, \varphi_2, \ldots, \varphi_k \ldots$ converges to zero in $D(\Omega)$ if all these functions vanish outside of certain fixed compact region, which is the same for all of them, and converge uniformly to zero (in the "usual" sense) together with their derivatives of any order.

Definition 4
We shall say that $\{\varphi_k\}|_{k\in N}$ converges to φ in $D(\Omega)$ if $\{\varphi_k - \varphi\}|_{k\in N}$ converges to zero in $D(\Omega)$.

Some notations:

- $K \subset\subset \Omega$ means $K \subset \Omega$ and K is compact.
- $\alpha = (\alpha_1, \ldots, \alpha_d) \in N_0^d$

$$\nabla^\alpha \varphi = \underbrace{\frac{\partial}{\partial x_1}\cdots\frac{\partial}{\partial x_1}}_{\alpha_1}\underbrace{\frac{\partial}{\partial x_2}\cdots\frac{\partial}{\partial x_2}}_{\alpha_2}\cdots\underbrace{\frac{\partial}{\partial x_d}\cdots\frac{\partial}{\partial x_d}}_{\alpha_d}\varphi$$

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_d} \end{pmatrix} \text{ is the Nabla operator}$$

For example,

$$\frac{\partial^3}{\partial x_1^2 \partial x_2} = \nabla^{(2,1,0,\ldots,0)}\varphi$$

The order of ∇^α is $\sum_{i=1}^d \alpha_i = |\alpha|$, and

$$\nabla^\alpha \varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\ldots\partial x_d^{\alpha_d}}$$

Definition 5
New Definition of Convergence in $D(\Omega)$:
$\{\varphi_k\}_{k\in N}$ converges to zero in $D(\Omega)$ iff:

1. *$\exists K \subset\subset \Omega$ such that $supp\, \varphi_k \subset K$ $\forall k \in N$*
2. *$\|\nabla^\alpha \varphi_k\|_{\infty,\Omega} \longrightarrow 0$ $\forall \alpha \in N_0^d$, if $k \longrightarrow \infty$*

1.2.2 Examples

Let's give us some examples of convergent sequences in our topolology:

Example
The sequence $\varphi_h(x) := \frac{1}{h}\left(\varphi(x+he_i) - \varphi(x)\right)$, $h = 1/k$ *converges to* $\frac{\partial}{\partial x_i}\varphi$ *in our topology.*
Proof is following:
Take $K = supp\,\varphi + B_2(0) := \{x+b | x \in supp\,\varphi,\, b \in B_2(0)\}$. *Then* $supp\,\varphi_k \subset \bar{K}$ *for all* $h = 1/k$.

$$\left\|\nabla^\alpha \varphi_k - \frac{\partial}{\partial x_i}\nabla^\alpha\varphi\right\|_\infty =$$

$$= \sup_{x\in\mathbb{R}^d}\left|\frac{1}{h}\left(\nabla^\alpha\varphi(x+he_i) - \nabla^\alpha\varphi(x)\right) - \frac{\partial\nabla^\alpha\varphi}{\partial x_i}(x)\right|.$$

Using Taylor's series, we can continue:

$$\left\|\nabla^\alpha \varphi_k - \frac{\partial}{\partial x_i}\nabla^\alpha\varphi\right\|_\infty \le \sup_{x\in\mathbb{R}^d}\left|\frac{\partial^2}{\partial x_i^2}\nabla^\alpha\varphi(x+\theta he_i)\right|\frac{h}{2},$$

where $\theta \in [0,1]$. *Now we can see that*

$$\left\|\nabla^\alpha \varphi_k - \frac{\partial}{\partial x_i}\nabla^\alpha\varphi\right\|_\infty \longrightarrow 0 \text{ when } k \longrightarrow \infty.$$

Example
It would be nice to mention, that sequence

$$\varphi_k(x) = \frac{1}{k}\varphi\left(\frac{x}{k}\right)$$

converges uniformly, but not in $D(\mathbb{R})$, *because diam supp* φ_k *goes to infinity, when* $k \longrightarrow \infty$.

1.3 More Test Functions

Here we'll show some methods, how to get new test functions, if we know one. Let's take, e.g., function $f(x)$ from page 18:

$$f(x) := \begin{cases} \exp\left(-\frac{1}{1-|x|^2}\right) & \text{, if } |x| < 1 \\ 0 & \text{, else} \end{cases} \quad \text{where } x \in \mathbb{R}^d.$$

Now define

$$m(x) := \frac{f(x)}{\int_{\mathbb{R}^d} f(y)\,dy}$$

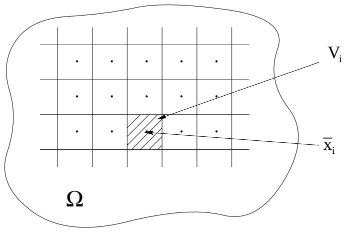


Figure 1.3: “Grid points” and “volumes”

Then there exists $m \in D(\mathbb{R}^d)$ such that $m \geq 0$, $supp\, m = \overline{B_1(0)}$, $m(x) = m(-x)$, and $\int_{\mathbb{R}^d} m(x)\, dx = 1$. We can obtain more test functions starting from them by

- **Scaling:**
$$m_\varepsilon(x) := \frac{1}{\varepsilon^d} m\left(\frac{x}{\varepsilon}\right), \varepsilon > 0.$$
Still $m_\varepsilon \geq 0\,, m_\varepsilon(x) = m_\varepsilon(-x)$ and $\int_{\mathbb{R}^d} m_\varepsilon(x)\, dx = \int_{\mathbb{R}^d} m(y)\, dy = 1$ (the substitution $y = x/\varepsilon$ was done), but $supp\, m_\varepsilon = \overline{B_\varepsilon(0)}$. So, our new object is a test function.

- **Shifting:** $m_\varepsilon(x - \bar{x})$ is again test function.

- **Linear combinations:** $\sum_{i=1}^n \alpha_i m_\varepsilon(x - \bar{x}_i)$ $,\alpha_i \in \mathbb{R}$ and $\bar{x}_i$ - some “grid points”. These linear combinations can be very useful - consider some smooth function $g(x)$. Then we know, that linear combination with coefficients $\alpha_i = g(\bar{x}_i)V_i$ is also test function. If V_i is some kind of “volume” (see figure 1.3), then such linear combination
$$\sum_{i=1}^n m_\varepsilon(x - \bar{x}_i) g(\bar{x}_i) V_i$$
converges to integral (1.1), when $n \longrightarrow \infty$.

- **Convolutions:**
$$\int_{\mathbb{R}^d} m_\varepsilon(x - \bar{x}) g(\bar{x})\, d\bar{x}. \tag{1.1}$$

Notation:
$$L^1_{loc} := \left\{ f : \Omega \to \mathbb{C} \,|\, f \text{ - measurable, } \int_K |f(x)|\, dx < \infty\ \forall K \subset\subset \Omega \right\}.$$

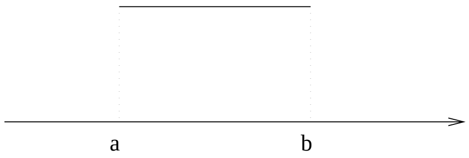


Figure 1.4: Indicator function $f(x) = 1_{[a,b]}$

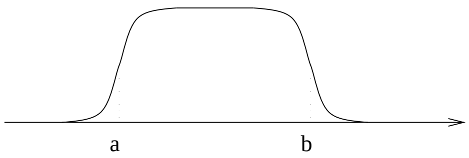


Figure 1.5: Function $f_\varepsilon(x)$

Remark
$L^p(\Omega) \subset L^1_{loc}(\Omega)\,,\ 1 \le p \le \infty$

Proof: *Let* $f(x) \in L^p$, *then*

$$\int_K |f(x)|\,dx = \int_K 1_K(x)|f(x)|\,dx$$

hence, using Holder's inequality we have

$$\int_K |f(x)|\,dx \le ||1_K||_{L^q}||f||_{L^p} = vol(K)^{1/q}||f||_{L^p} \le \infty$$

for p, q *s.t.* $\frac{1}{p} + \frac{1}{q} = 1$ *(since indicator function* $1_K(x) \in L^q$ *and* $|f(x)| \in L^p$*). From this follows that* $f(x) \in L^1_{loc}$.
For $f \in L^1_{loc}(\mathbb{R}^d)$, *the convolution*

$$f_\varepsilon(x) := \int_{\mathbb{R}^d} m_\varepsilon(x - \bar{x}) f(\bar{x})\,d\bar{x}$$

is well definied. In fact, $f_\varepsilon \in C^\infty(\mathbb{R}^d)$, *and if* $supp\, f$ *is compact then* $f_\varepsilon \in C_0^\infty(\mathbb{R}^d)$.

Example
Let's take indicator function (see figure 1.4):

$$f(x) = 1_{[a,b]}(x)$$

$f \in L^1_{loc}(\mathbb{R})$, *because* $f \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$.

Note:

$$f_\varepsilon(x) = \begin{cases} 0 & if \quad x < a - \varepsilon\,, x > b + \varepsilon \\ 1 & if \quad a + \varepsilon < x < b - \varepsilon \end{cases}$$

(See figure 1.5.)

Chapter 2

DISTRIBUTION

2.1 SOME DEFINITIONS

2.1.1 Definition of Functional

A functional on a linear space E is a mapping $f : E \to C$ which assigns to each member ϕ of E to a certain complex number .That is ,image of $\phi \in E$ under f is $f(\phi) \in C$.

2.1.2 Definition of Linearity

f is said to be a Linear functional on E if and only if for any two numbers ϕ_1, ϕ_2 of E and any scalars a_1, a_2 we have:

$f(a_1\phi_1 + a_2\phi_2) = a_1 f(\phi_1) + a_2 f(\phi_2)$.

2.1.3 Definition of continuous Functionl

f is said to be a continuous functional on E if and only if whenever a sequence $\{\phi_n\}_{n\in N}$ converges to 0 in E ,the corresponding sequence $f(\phi_n)$ of complex number converges to 0.

2.1.4 Definition of Distribution

Let Ω be an open subset of R^d. $D(\Omega)$ is the space of infinitely differentiable functions in Ω and having compact support in Ω.
We write $D'(\Omega)$ is the set of continuous linear functionals on $D(\Omega)$.
The elements of $D'(\Omega)$ are called distributions or generalized functions.

Remark1

If T is a distribution :
a) Linear functional implies that
$T(\alpha\Psi + \beta\phi) = \alpha(T(\Psi)) + \beta(T(\phi))\ \forall\alpha, \beta \in C, \forall\phi, \Psi \in D(\Omega)$.
b) Continuity in $D(\Omega)$ implies:
As $\phi_k \to 0$ in $D(\Omega)$ as k $\to \infty$
$\Rightarrow T(\phi_k) \to 0$ as $k \to \infty$.

Note:

$\phi_k \to \phi$ in $D(\Omega)$ as $k \to \infty$
$\Rightarrow \phi_k - \phi \to 0$ in $D(\Omega)$ as k $\to \infty$. (By definition)
$\Rightarrow T(\phi_k - \phi) \to 0$ as $k \to \infty$ (By definition of continuity)
$\Rightarrow T(\phi_k) \to T(\phi)$(By definition of Linearity)

2.2 Examples of Distribution

2.2.1 Regular distribution

we say that :
$f \in L^1_{loc}(\Omega) = \{f : \Omega \to C/$ f is measurable,$\int_k \mid f \mid < \infty$: k compact subset of $\Omega\}$
For example :$f(x) = \frac{1}{x} \in L^1_{loc}(0, \infty)$,

$$g(x) = \begin{cases} \frac{1}{x} & x > 0 \\ 0 & x \geq 0 \end{cases}$$

But,g(x) does not belong to $L^1_{loc}(R)$.
We consider f is locally integrable function.
Define $T_f(\phi) = \int_\Omega f(x)\phi(x)dx$, where $\phi \in D(\Omega)$.
Now $T_f(\phi)$ is linear:

$$\begin{aligned} T_f(\alpha\phi + \beta\Psi) &= \int_\Omega (\alpha\phi + \beta\Psi) f(x)dx \\ &= \int_\Omega \alpha\phi f(x)dx + \int_\Omega \beta\Psi f(x)dx \\ &= \alpha\int_\Omega \phi f(x) + \beta\int_\Omega \Psi f(x)dx \\ &= \alpha T_f\phi + \beta T_f\Psi \end{aligned}$$

Now $T_f(\phi)$ is continuous:
Let $\phi_K \to 0$ in $D(\Omega)$ as $k \to \infty$.
$\Rightarrow \exists$ a compact set k $\subset \Omega$ such that supp $\phi_k \subset k, \forall k \in N$ and ϕ_k and all its derivatives converges to 0 uniformly on k.
That is $\exists k \subset\subset \Omega$, supp $\phi_k \subset k$ and $\| \bigtriangledown^\alpha \phi_k\|_\infty \to 0$ as$k \to \infty : \forall\alpha$
Now

$$\begin{aligned}
\mid T_f(\phi_k) \mid \quad &= \mid \int_\Omega f(x)\phi_k(x)dx \mid \\
&\leq \int_\Omega \mid f(x) \mid\mid \phi_k(x) \mid dx \\
&\leq \int_\Omega \mid f(x) \mid sup \mid \phi_k(x) \mid dx \\
&= \|\phi_k\|_\infty \int_k \mid f(x) \mid dx
\end{aligned}$$

But fis locally integrable$\Rightarrow \int_k \mid f(x) \mid dx < \infty$
But $\|\phi_k\|_\infty \to 0$ as $k \to \infty$
Therefore ,$\mid T_f(\phi_k) \mid \to 0$ as$k \to \infty$
Therefore,$\phi_k \to 0$ as $k \to \infty$
$\Rightarrow T_f(\phi_k) \to 0$ as $k \to \infty$
Therefore, $T_f(\phi)$ is continuous
Therefore ,$T_f(\phi)$ is a continuous linear functional on $D(\Omega)$
$\Rightarrow T_f \in D'(\Omega)$

Remark

(1) Any locally integrable function f corresponds a distribution $T_f(\phi)$. Such a distribution is said to be regular.
(2) If two locally integrable functions are equal almost everywere,the corresponding regular distributions are equal.
(3) f is uniquely determined by T_f
Proof: Let

$$\begin{aligned}
T_f = T_g \quad &\Rightarrow T_f(\phi) = T_g(\phi), \forall\phi \in D(\Omega) \\
&\Rightarrow \int_\Omega (f-g)\phi dx = 0, \forall\phi \in D(\Omega) \\
&\Rightarrow f - g = 0 (For the case f and g are continuous)
\end{aligned}$$

(For the general case see YOSIDA ,Functional Analysis)
We shall prove this by contradiction

Let $f - g \neq 0$
That is

$$f - g = h \neq 0$$

$$\begin{aligned}
&\Rightarrow \exists \overline{x} \in \Omega; h(\overline{x}) > 0(or < o) \\
&\Rightarrow h(x) > 0; \forall x \in B_\epsilon(\overline{x}) for some \epsilon (Continuity of h) \\
&\Rightarrow \int h(x) m_\epsilon(x - \overline{x}) dx > 0 or \geq 0 (m_\epsilon(x - \overline{x}) > o) \\
&\Rightarrow T_h(m_\epsilon) > 0 : but T_h(\phi) = 0
\end{aligned}$$

$which contradicts$

Therefore,our supposition os wrong

$$\begin{aligned}
&\Rightarrow h \neq 0 \\
&\Rightarrow f - g = 0 \\
&\Rightarrow f = g
\end{aligned}$$

Note;
$L^1_{loc} \subset D'(\Omega)$ $(Because f \in D'(\Omega)$ means $T_f \in D'(\Omega))$
That is f andT_f are identified
That is we can interpret locally integarable function as a Distributions.
$D'(\Omega) \backslash L^1_{loc}(\Omega)$ are singular distributions$(Because D'(\omega)$ contains more objects)

2.3 Dirac delta function

Now$\delta \in D'(R^d)$ is defined by
$\delta(\phi) = \phi(0); \forall \phi \in R^d$
Now, We shall prove that :it is really a distribuion
(1) $\delta(\phi)$ is linear:

$$\begin{aligned}
\delta(\alpha\phi + \beta\Psi) & \\
&= (\alpha\phi + \beta\Psi)(0) \\
&= \alpha\phi(0) + \beta\Psi(0) \\
&= \alpha\delta(0) + \beta\delta(0) \\
&= \alpha\delta(\phi) + \beta\delta(\phi)
\end{aligned}$$

(2) $\delta(\phi)$ is continuous
Let $\phi_n \to 0$ in $D(R^d)$

Now

$$\begin{aligned} | \delta(\phi_n) | \\ &= | \phi_n(0) | \\ &\leq sup | \phi_n(0) | \\ &= \| \phi_n \|_\infty \to 0 as n \to \infty \end{aligned}$$

Therefore
$\phi_n \to o$
$\Rightarrow \delta(\phi_n) \to 0 as n \to \infty$
$\Rightarrow \delta(\phi)$ is continuous.
Therefore,$\delta(\phi)$is distribution.
It will not represent a regular distribution ,since there is no locally integrable function h(x) such that
$\delta(\phi) = \int_{R^d} h(x)\phi(x)dx$
Note;
More general, for any $\overline{x} \in R^d$, We define
$\delta_{\overline{x}} \in D'(R^d)$ by $\delta_{\overline{x}}(\phi) = \phi(\overline{x})$

2.4 Measures as Distributions

Let μ be a positive Borel measure on R^d. Then every continuous function of compact support is integrable with respect to μ and the map $\phi \to \int_{R^d} \phi(x)d(\mu x) = \mu(\phi)$ defines a linear functional on$C_0^0(R^d)$ which is positive in the sense that $\mu(x) \geq 0$ for every real valued functions$\phi \in C_0^0(R^d)$ which is nonnegative onR^d such a functional is called a Radon measure onR^d;
Now,μ : Positive radon Measure on R^d.
Define $s_\mu(\phi) = \int_{R^d} \phi(x)d\mu(x) : \phi \in D(R^d)$
We shall show that $s_\mu(\phi)$ is a distribution:
(1) $s_\mu(\phi)$ is linear:

$$\begin{aligned} Now, S_\mu(\alpha\phi + \beta\Psi) \\ &= \int_{R^d} (\alpha\phi + \beta\Psi)d\mu(x) \\ &= \alpha \int_{R^d} \phi d\mu(x) + \beta \int_{R^d} \Psi d\mu(x) \\ &= \alpha s_\mu \phi + \beta s_\mu \Psi \end{aligned}$$

Therefore $s_\mu(\phi)$ is linear
(2)$s_\mu(\phi)$ is continuous

let$\phi_k \to 0$ as k $\to \infty$ in D(R^d)
$\Rightarrow \exists k \subset\subset R^d$; supp$\phi_k \subset k$;$\forall k \in N$

$$\begin{aligned}
\Rightarrow \mid s_\mu(\phi_k) \mid & \\
&= \mid \int_{R^d} \phi_k(x) d\mu(x) \mid \\
&= \mid \int_k \phi_k d\mu(x) \mid \\
&\leq \|\phi_k\|_\infty \int_k d\mu \\
&= \| \phi_k \|_\infty \mu(k) \to 0ask \to \infty
\end{aligned}$$

(But,μ is a Radon Measure ,$\mu(k) < \infty$)
Therefore,
$\phi_k \to 0$ inR^d
$\Rightarrow s_\mu(\phi_k) \to 0inR^d$
$s_\mu(\phi)$ is continuous
Note:
let

$$\Delta(A) = \begin{cases} 1 & x \in A \\ 0 & xnotinA \end{cases}$$

is a Positive readon measure
Therefore, s$_\Delta(\phi) = \int \phi(x) d\Delta(x) = \phi(0) = \delta(\phi)$
$\Rightarrow \delta = s_\Delta$
$\Rightarrow \delta$ can be identified with a positive Radon measure

2.5 vectorspace structure

Given S,T$\in D'(\Omega)$ and $\alpha, \beta \in$C,
we define
$(\alpha S + \beta T)(\phi) = \alpha s(\phi) + \beta T(\phi), \forall \phi \in D(\Omega)$
Now we shall prove that $\alpha S + \beta T \in D'(\Omega)$
1)Linearity:

$$\begin{aligned}
(\alpha S + \beta T)(a\phi + b\Psi) &= \alpha S(a\phi + b\Psi) + \beta T(a\phi + b\Psi) \\
&= a(\alpha s(\phi) + \beta T(\phi)) + b(\alpha s(\Psi) + \beta T(\Psi)) \\
&= a(\alpha s + \beta T)(\phi) + b(\alpha s + \beta T)(\Psi)
\end{aligned}$$

2)Continuity:
Let $\phi_i \to 0$ in $D(\Omega)$

$$\Rightarrow s(\phi_i) and T(\phi_i) \to 0$$
$$\Rightarrow (\alpha s + \beta T)(\phi_i) \to 0$$

Now, we shall find the generalization of linear combinations of functions
Let f,g$\in L^1_{loc}(\Omega)$,

$$\begin{aligned}(\alpha T_f + \beta T_g)(\phi)&\\ &= \alpha T_f(\phi) + \beta T_g(\phi)\\ &= \alpha \int f\phi dx + \beta \int g\phi dx\\ &= \int (\alpha f + \beta g)\phi dx = T_{(\alpha f+\beta g)}(\phi), \forall \phi \in D(\Omega)\end{aligned}$$

That is
$\alpha T_f + \beta T_g = T_{\alpha f+\beta g}$
Note:$D'(\omega) X D(\omega)$ $\to$C
That is ,$(T, \phi) \to T(\phi)$ is a bilinear operation denote by a dual pairing
$< T, \phi >= T(\phi)$ .Now$< ., . >$ is linear in each argument.
Assume,ϕ depends on two variables x,y and
x$\to$ ϕ(x,y) $\in$D'(Ω) $\forall$y$\in$Y.
To stress that T in $< T, \phi(., y) >$ acts on the x variable, we write $< T(x), \Phi(x, y) >_x$
Attention:
T(x) does not mean the value of T at x which makes no sense for general T $\in$ D'(Ω)
It just means T acts on the x-Variable.
Side effect of new notation:
Define also $<, > : l^1_{loc}(\Omega) X D(\Omega) \to C$,
That is (f,ϕ) $\to < f, \phi >= \int f\phi dx$
then, $< f, \phi >=< T_f, \phi >$.

Chapter 3

Extension of the differentiability concept

3.1 practical example

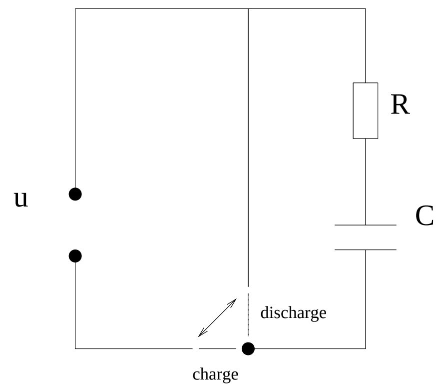


Figure 3.1: eletric charge in the capacitor

equation for eletric charge in the capacitor

voltage at capacitor: $u_C = \frac{Q}{C}$, where C is capacity.

voltage at resistor : $u_R = RI$

Note that $I = \dot{Q}$,

during charge phase: $u = u_C + u_R$

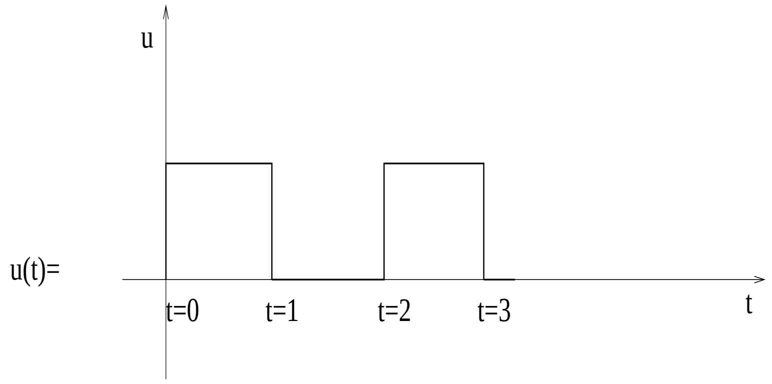


Figure 3.2: voltage u

during discharge phase: $0 = u_C + u_R$

To discribe sequence of charge/discharge, we use the equation,

$u(t) = u_C + u_R = \frac{Q}{C} + R\dot{Q}$,

furthurmore we can get the equation about $Q(t)$

$$\dot{Q} = \frac{1}{R}u - \frac{1}{RC}Q,$$

with initial condition $Q(0) = 0$ (it means no initial charge). From this equation we note that $Q \notin C^1(R)$, since u has jumps. In this case u is discontinuous, the classical ODE theory fails, then we have to study the existence , uniqueness , ... , ?

The ODE theory with discontinuous is wanted, that is , we need generalized differentiability which yields, for example,

$$\frac{d}{dt}(corner) = jump$$

In the sequel, differentiation of distributions will be introduced, and after that we will consider ODEs with distributions on right hand side.

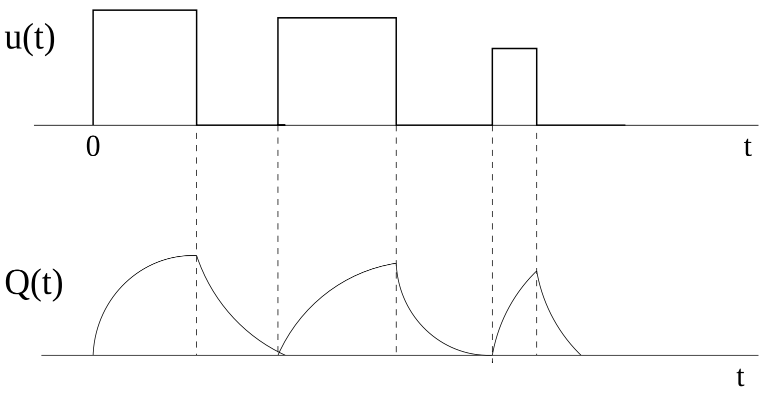


Figure 3.3: the relation of voltage u and eletric charge Q

3.2 Differentiation

3.2.1 Differentiation on 1-D space

For any $f \in C^1(\Omega)$, take $\varphi \in D(\Omega)$ such that $supp\varphi \subset\subset (-R, R)$ for some $R > 0$. Define

$$< \tfrac{d}{dx}T_f, \varphi >:=< T_{\frac{d}{dx}f}, \varphi >$$

From the fundamental theorem of calculus, we can have

$< T_{\frac{d}{dx}f}, \varphi >=< \frac{d}{dx}f, \varphi >$

$$= \int f'\varphi dx = \int_{-R}^{R} f'\varphi dx$$

$$= \int_{-R}^{R}(f\varphi)' - f\varphi' dx$$

$$= f\varphi|_{-R}^{R} - \int_{-R}^{R} f\varphi' dx \qquad (\varphi(R) = \varphi(-R) = 0)$$

$$= -\int f\varphi' dx = - < f, \tfrac{d}{dx}\varphi >$$

$$= - < T_f, \tfrac{d}{dx}\varphi >$$

This give us a meaning for the derivative of a generalized function, generally, for any $S \in D'(\Omega)$, set

$$< \tfrac{d}{dx}S, \varphi >:= - < S, \tfrac{d}{dx}\varphi > \qquad \forall\varphi \in D(\Omega)$$

This defines a distribution $\frac{d}{dx}S$ because $\frac{d}{dx} : D(\Omega) \rightarrow D(\Omega)$ is linear and continuous:

1. since $\varphi \in D(\Omega)$, we can get $\varphi \in C^{\infty}$ and $supp\varphi' \subset supp\varphi$, so that $supp\varphi'$ is compact, then we can get $- < S, \varphi' >$ is well defined, that means $\varphi \in D(\Omega)$.
2. mapping $\varphi \rightarrow -\frac{d}{dx}\varphi$ is linear, so mapping $\varphi \rightarrow - < S, \varphi' >$ is linear.
3. for sequence $(\varphi_i)_{i \int N}$, $\varphi_i \in D(\Omega)$ for any $i \in N$, let $\varphi_i \rightarrow 0$ while $i \rightarrow \infty$, we can get $supp\varphi_i \subset K \subset\subset \Omega$, $\forall i$, furthermore we get $supp\varphi_i^{'} \subset K, \forall i$.
   $\| (\frac{d}{dx})^{\alpha} \frac{d}{dx}\varphi_i \|_{\infty} = \| (\frac{d}{dx})^{\alpha+1}\varphi_i \|_{\infty} \rightarrow 0$
   while $i \rightarrow \infty$, $\forall \alpha \in N_0$, $N_0 = \{1, 2, ..., N\}$
   then we have mapping $\varphi \rightarrow - < S, \varphi' >$ is continuous.

3.2.2 Generalization to $d > 1$

for any $f \in C'(\Omega)$, define

$< \frac{\partial}{\partial x_i}T_f, \varphi > := < \frac{\partial f}{\partial x_i}, \varphi >$

$= \int \frac{\partial f}{\partial x_i}\varphi dx$

$= \int \frac{\partial}{\partial x_i}(f\varphi) - f\frac{\partial \varphi}{\partial x_i}dx$

$= \int_{B_R} div(f\varphi e_i)dx - \int f\frac{\partial \varphi}{\partial x_i}dx \qquad (\text{supp}\varphi \subset\subset B_R)$

$= \int_{\partial B_R} f\varphi n_i d\sigma - < f, \frac{\partial \varphi}{\partial x_i} > \qquad (\text{Gauss Theorem})$

$= - < T_f, \frac{\partial}{\partial x_i}\varphi >$

<u>Generalize</u>: $S \in D'(\Omega)$,

$< \frac{\partial}{\partial x_i}s, \varphi > := - < S, \frac{\partial \varphi}{\partial x_i} >$

<u>higher</u> derivatives: (iterate)

$< \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_i}S, \varphi >$

$= - < \frac{\partial}{\partial x_i} S, \frac{\partial}{\partial x_k} \varphi >$

$=< S, \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} \varphi >$

$=< S, \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_i} \varphi >$

$= - < \frac{\partial}{\partial x_k} S, \frac{\partial}{\partial x_i} \varphi >$

$=< \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} S, \varphi >$

so derivatives of arbitrary order exist and order of differentiation can be exchanged.

$\nabla^\alpha S := (\frac{\partial}{\partial x_1})^{\alpha_1} \ldots (\frac{\partial}{\partial x_d})^{\alpha_d} S$ make senses! for any $\alpha \in N_0^d$ and

$$< \nabla^\alpha S, \varphi >= (-1)^{|\alpha|} < S, \nabla^\alpha \varphi >$$

Chapter 4

Examples

4.1 Smooth functions

let $f \in C'(\Re)$
By construction : $(T_f)^{'} = T_{f'} \longrightarrow$ nothing new

4.2 Functions with "corners"

Consider the function $f(x) = |x|$

$$\begin{aligned}
< \frac{d}{dx} T_f, \varphi > \quad &= - < T_f, \varphi^{'} > = - \int_{\infty}^{\infty} |x| \varphi^{'} dx \\
&= - \int_0^{\infty} x \varphi^{'}(x) dx + \int_{-\infty}^{0} x \varphi^{'}(x) dx
\end{aligned}$$

(by inititial value problem)

$$\begin{aligned}
&= \underbrace{-x\varphi(x)|_0^{\infty}}_{=0} + \int_0^{\infty} x\varphi(x)dx + \underbrace{-x\varphi|_{-\infty}^{0}}_{=0} - \int_{-\infty}^{0} x\varphi(x)dx \\
&= < H(x), \varphi(x) >_x + < -H(-x), \varphi(x) >_x \\
&= < H(x) - H(-x), \varphi(x) >_x
\end{aligned}$$

With sloppy notation : $|x|' = H(x) - H(-x)$
One can check in general : height of jump = right slope-left slope

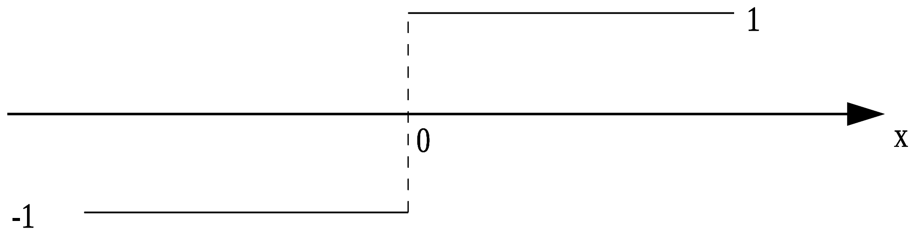


Figure 4.1: The graph of $|x|' = H(x) - H(-x)$

4.3 Functions with jumps

Consider standard jump function H(x) (Heaviside function)

$$< \frac{d}{dx}T_H, \varphi > = < H, \varphi' > = -\int_0^\infty \varphi(x)' dx = -\varphi|_0^\infty = \varphi(0) = < \delta, \varphi >$$

With sloppy notation $H' = \delta$
Hence ‘ Jumps leads to deltas’
One can check in general : ”Strength ” (pre-factor) of delta= right value-left value
Hence : $|x|'' = 2\delta$ ($|x|'$ jumps from -1 to 1)

4.4 Functions with weakly singular derivatives

Consider the function

$$f(x) = 2\sqrt{x_+} \quad where \quad x_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Classically $f'(x) = \frac{1}{\sqrt{x}} \;\; for \;\; x > 0 \;\; and \;\; f'(x) = 0 \;\; for \;\; x < 0$
Therefore f(x) is not differentiable at x=0

$$\begin{aligned} < \frac{d}{dx}T_f, \varphi > \quad &= < f, \varphi' > = -\int_0^\infty 2\sqrt{x}\varphi' dx \\ &= -2\sqrt{x}\varphi|_0^\infty + \int_0^\infty \frac{\varphi(x)}{\sqrt{x}} dx \\ &= < \frac{1}{\sqrt{x_+}}, \varphi(x) >_x \end{aligned}$$

$\Longrightarrow 2\sqrt{x_+}' = \frac{1}{\sqrt{x_+}}$

4.5 Functions with strongly singular derivatives

$$f(x) = \frac{1}{x_+^\alpha} \quad where \quad \alpha \in (0,1)$$

note $f \in L'_{loc}\Re$ but the classical derivative

$$f'(x) = \begin{cases} \frac{-\alpha}{x^{\alpha+1}} & x > 0 \\ 0 \quad x > 0 \end{cases} \qquad is\ not\ integrable$$

What is weak derivative ?

$$< \frac{d}{dx}T_f, \varphi >= - < f, \varphi' >= - \int_0^\infty \frac{\varphi'(x)}{x^\alpha}dx$$

Should resemble $\frac{-\alpha}{x_+^{\alpha+1}}$ though
Then define $\frac{1}{x_+^{\alpha+1}}$ appropriately
a) Recognize to recover classical $\frac{1}{x_+^{\alpha+1}}$

$$- \int_0^\infty \frac{\varphi'(x)}{x^\alpha}dx \qquad \underbrace{=}_{\text{ibp has problem at x=0}} \lim_{\epsilon\to 0} - \int_\epsilon^\infty \frac{\varphi'(x)}{x^\alpha}dx$$

$$= \lim_{\epsilon\to 0}\left(\frac{\varphi(\epsilon)}{\epsilon^\alpha} + \int_\epsilon^\infty \frac{-\alpha\varphi(x)}{x^{\alpha+1}}dx\right)$$

Define

$$\psi(x) = \begin{cases} \frac{\varphi(x)-\varphi(0)}{x} & x \neq 0 \\ \varphi'(0) & x = 0 \end{cases}$$

then

$$\psi \in C(\Re) \text{ becuase} \lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x-0} = \varphi'(0)$$

note

$$\varphi(x) = \varphi(0) + x\psi(x) \forall x \in \Re$$

and

$$\frac{\varphi(\epsilon)}{\epsilon^{\alpha}} = \frac{\varphi(0)}{\epsilon^{\alpha}} + \underbrace{\epsilon^{1-\alpha}\psi(\epsilon)}_{\to 0 as \epsilon \to 0}$$

Thus

$$= -\int_0^{\infty} \frac{\varphi^{'}(x)}{x^{\alpha}}dx = -\alpha \lim_{\epsilon\to 0}(\int_{\epsilon}^{\infty} \frac{\varphi^{'}(x)}{x^{\alpha}}dx - \underbrace{\frac{\varphi(0)}{\alpha\epsilon^{\alpha}}}_{\text{singular part of integral is subtracted}}$$

b) "Finite part" is defined in the sense of Hadamard)

$$FP\int_{\epsilon}^{\infty} \frac{\varphi^{'}(x)}{x^{\alpha}}dx := \lim_{\epsilon\to 0}(\int_{\epsilon}^{\infty} \frac{\varphi^{'}(x)}{x^{\alpha}}dx - \frac{\varphi(0)}{\alpha\epsilon^{\alpha}})$$

and the "pseudo function " $PF\frac{1}{x_+^{\alpha+1}} \in D^{'}(\Re)\ isdefinedby$

$$< PF\frac{1}{x_+^{\alpha+1}}, \varphi >:= FP\int_{\epsilon}^{\infty} \frac{\varphi^{'}(x)}{x^{\alpha}}dx$$

then

$$\frac{d}{dx}T_{\frac{1}{x_+^{\alpha}}} = -\alpha PF\frac{1}{x_+^{\alpha+1}}$$

Another example of this kind: $f(x) = ln(x)$ note that $f \in L^{'}_{loc}(\Re)$

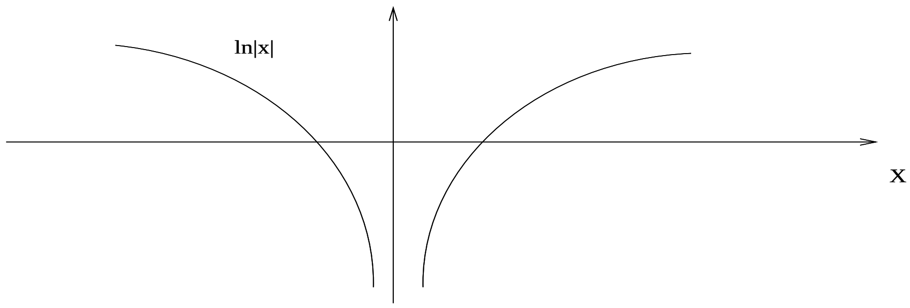


Figure 4.2: The graph of $f(x) = ln(|x|)$

The classical derivative : $f^{'}(x) = \frac{1}{x}$ for $x \neq 0 is\ not\ integrable\ at x = 0$

Distributional derivative ?

$$\begin{aligned} < \frac{d}{dx}T_f, \varphi > &= -\int_0^\infty ln x\varphi^{'}(x)dx - \int_{-\infty}^0 ln(-x)\varphi^{'}(x)dx \\ &= \lim_{\epsilon\to 0}(ln(\epsilon)\varphi(\epsilon) + \int_\epsilon^\infty \frac{\varphi(x)}{x}dx - ln(\epsilon)\varphi(-\epsilon) + \int_0^{-\infty} \frac{\varphi(x)}{x}dx) \end{aligned}$$

use again $\varphi(x) = \varphi(0) + x\psi(x)$
Then $\pm ln(\epsilon)\varphi(\pm\epsilon) = \pm ln(\epsilon)\varphi(0) + \underbrace{\epsilon ln(\epsilon)\varphi(-\epsilon)}_{\to 0 \ as \ \epsilon\to 0}$

Hence

$$< \frac{d}{dx}T_f, \varphi >= \lim_{\epsilon\to 0}\int_{|x|>\epsilon} \frac{\varphi(x)}{x}dx$$

Define

$$PV\int_{-\infty}^\infty \frac{\varphi(x)}{x}dx = \int_{|x|>\epsilon} \frac{\varphi(x)}{x}dx$$

"Cauchy principal value of $\int_{-\infty}^\infty \frac{\varphi(x)}{x}dx$"
and

$$< PF\frac{1}{x}, \varphi >= PV\int_{-\infty}^\infty \frac{\varphi(x)}{x}dx$$

and

$$\frac{d}{dx}ln(|x|) = PF\frac{1}{x} \qquad \text{in } D'(R)$$

Note :

$$f \in L_{loc}^{'}(R) \qquad < f, \varphi >:= \int_{-\infty}^\infty f(x)\varphi(x)dx$$

We want : similar notation for $f \notin L_{loc}^{'}$ if possible
Here :

$$< PF\frac{1}{x}, \varphi >= PV\int_{-\infty}^\infty \frac{\varphi(x)}{x}dx$$

or :

$$< PF\frac{1}{x_+^{\frac{3}{2}}}, \varphi >= FP\int_{-\infty}^\infty \frac{\varphi(x)}{x_+^{\frac{3}{2}}}dx$$

Chapter 5

Ordinary Differential Equations

5.1 A simple equation

$I \subset \mathbb{R}$: open interval

$u' = 0$ in $D'(I) \Leftrightarrow u$ is constant

$(u = T_f$ with $f(x) = c \forall x \in I)$

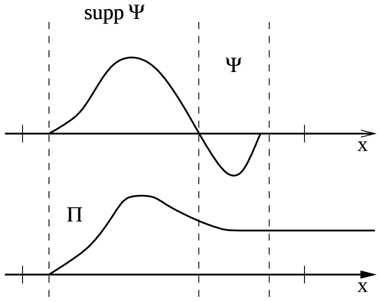
<u>Proof:</u>

" $\Longleftarrow$ " if u is constant then $u = T_f$ with $f \in C^\infty(I)$, $u' = T_f = T_0 = 0$

" $\Longrightarrow$ " Assume $u' = 0$

$\overset{Def.}{\Longrightarrow} -\langle u, \varphi' \rangle = 0 \ \forall \varphi \in D(I)$

Let	$\psi \in D(I)$	define	$\Pi(x) := \int_{-\infty}^{x} \psi(x) dy$
Then	$\Pi \in C^\infty(\mathbb{R})$		
but	$Pi \in C_0^\infty(\mathbb{R})$	only if	$\langle 1, \psi \rangle = \int_{-\infty}^{\infty} \psi(y) dy = 0$
Choose	$\bar{\psi} \in D(I)$	such that	$\langle 1, \bar{\psi} \rangle = 1$
then	$I(\psi - \langle 1, \bar{\psi} \rangle \bar{\psi}) = 0$		
and	$\varphi(x) := \int_{-\infty}^{\infty} \psi(y) - \langle 1, \bar{\psi} \rangle \bar{\psi}(y) dy$		is a test function
with	$\varphi' = \psi(x) - \langle 1, \bar{\psi} \rangle \bar{\psi}(x)$		
Hence	$\langle u, \psi \rangle - \langle 1, \psi \rangle \langle u, \bar{\psi} \rangle = 0 \quad \forall \psi \in D(I)$		
	$\langle u, \bar{\psi} \rangle = c$		



$\Pi = \int_{-\infty}^{\infty} \Psi(y)dy$

$$
\begin{array}{ll}
\Longrightarrow & \langle u, \psi\rangle = c\langle 1, \psi\rangle = \langle c, \psi\rangle \quad \forall\psi \\
\Longrightarrow & u = c
\end{array}
$$

■

5.2 A Linear Fisrt Order System

$m \in \mathbb{N}$, $A \in C^\infty(I)^{m\times m}$, $f \in C^0(I)^m$

Assume $u \in [D'(I)]^m$ satisfies

$(*) \qquad u' + Au = f$ in $(D'(I))^m$

Then $u \in C'(I)$ and $(*)$ is satisfied in the classical sense.

<u>Proof:</u>

From theory of ODEs:

There exist $0 \neq E \in (C^\infty(I))^{m\times m}$ such that

- $E' = EA$
- E invertible with $E^{-1} = (C^\infty(I))^{m\times m}$

Note:

$$
\begin{aligned}
(Eu)' &= E'u + Eu' \qquad \text{product rule in } D'(I) \\
&= EAu - EAu + Ef \\
&= Ef
\end{aligned}
$$

Let $F \in (C'(I))^m$ be a primitive of Ef in I
Then $(Eu - F)' = 0$ in $D\prime(I)$
$\Longrightarrow$ $Eu = F + c$
$\Longrightarrow$ $u = E^{-1}(F + c) \in C'(I)$

■

remember: need to know for smooth functions $(E, \dots)$ then generalization is "easy" (sometimes).

5.3

If $u \in D'(I)$ satisfies

$$u^{(m)} + a_{m-1}u^{(m-1)} + \cdots + a_0 u = f$$

with $a_i \in C^\infty(I)$ and $f \in C^0(I)$
then $u \in C^m(I)$ and the equation holds classically.

Proof:
Transform into systems of first order equations. Use previous result.

■

Example:

$v'' + av' + bv = c$
$u_1 := v$, $u_2 := v'$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}' = \begin{pmatrix} v \\ v' \end{pmatrix}' = \begin{pmatrix} v' \\ v'' \end{pmatrix} = \begin{pmatrix} u_2 \\ -au_2 - bu_1 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

$\mathbf{u}' + \mathbf{A}\mathbf{u} = \mathbf{f}$
where $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ b & a \end{pmatrix}$ and $\mathbf{f} = \begin{pmatrix} 0 \\ c \end{pmatrix}$

5.4 Equations with Singular Sources

Assume $S \in D'(I)$ then there exist $u \in D'(I)$ such that $u' = S$. Two solutions of $u' = S$ differ only by a constant.

Proof:

$\langle u, \varphi\rangle \overset{Def.}{:=} -\langle S(x), \int_{-\infty}^{x} \psi(y) - \langle 1, \psi\rangle\bar{\psi}(y)dy\rangle_x \ \langle 1, \bar{\psi}\rangle = 1$

Note:

$$\begin{aligned}\langle u', \varphi\rangle &= \langle u, \varphi'\rangle = \langle S(x), \int_{-\infty}^{x} \varphi'(y) - \langle 1, \varphi'\rangle\bar{\psi}(y)dy\rangle_x \\ &= \langle S(x), \varphi(x)\rangle_x \forall\varphi \in itD(I)\end{aligned}$$

since $\quad \langle 1, \varphi'\rangle = -\langle 1', \varphi\rangle = 0$
$\Longrightarrow \quad u' = S$
Assume$v' = S \Rightarrow (u-v)' = 0 \Rightarrow u - v = constant$

■

Example:

$u' = \delta$

$$\begin{aligned}\langle u, \psi\rangle &= -\int_{-\infty}^{0} \psi(y)dy + \langle 1, \psi\rangle \underbrace{\int^{0} \bar{\psi}(y)dy}_{c} \\ &= \langle c - H(-y), \psi(y)\rangle_y \\ &= \langle c - 1 + H(y), \psi(y)\rangle_y\end{aligned}$$

$$\begin{aligned}\Longrightarrow u &= H + \int_{-\infty}^{0} \bar{\psi}dy - 1 \\ \Longrightarrow u &= H \qquad \text{up to constants}\end{aligned}$$

Consider system:

$$u' + Au = S$$

with $A \in (C^\infty(I))^{m\times m}$ and $S \in ((D)'(I))^m$

As before, let $E \in C^\infty(I)$, $E = E' = EA$ and let $T \in (D^\infty(I))$ be a solution of

$$T' = ES$$

Define: $u := E^{-1}T$
then $(Eu)' = T' = ES$
and $(Eu)' = E'u + Eu' = EAu + Eu'$

multiplying with E^{-1} yields: $u' + Au = S$

Note: two solutions of $u' + Au = S$ differ by a solution of $w' + Aw = 0$ which is a classical solution.

Example:

$v'' + v = \delta D'(\mathbb{R})$

can be written as: $\mathbf{u}' + \mathbf{A}\mathbf{u} = \mathbf{S}$

$$\text{where } \mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{S} = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

Construction of E:

Solve: $\mathbf{w}' = \mathbf{A}^T\mathbf{w}, \mathbf{w}(0) = \mathbf{e}_i, i = 1, 2$. The solutions are $\mathbf{a}$ and $\mathbf{b}$

$$\text{Set } \mathbf{E} = \begin{pmatrix} \mathbf{a}^T \\ \mathbf{b}^T \end{pmatrix}$$

$$\text{Then } \mathbf{E}' = \begin{pmatrix} \mathbf{a}^{T\prime} \\ \mathbf{b}^{T\prime} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^T\mathbf{A} \\ \mathbf{b}^T\mathbf{A} \end{pmatrix} = EA$$

$$\text{Consider } \mathbf{w}' = \mathbf{A}^T\mathbf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Solutions are:

$$\mathbf{a} = \begin{pmatrix} -\cos x \\ \sin x \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix}$$

$$\mathbf{E} = \begin{pmatrix} -\cos x & \sin x \\ \sin x & \cos x \end{pmatrix}, \qquad E^{-1} = E$$

$$\mathbf{ES} = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} \delta \overset{trick}{=} \begin{pmatrix} \sin 0 \\ \cos 0 \end{pmatrix} \delta = \begin{pmatrix} 0 \\ \delta \end{pmatrix}$$

$$\text{Hence } \mathbf{T} = \begin{pmatrix} 0 \\ H \end{pmatrix} \qquad \text{satisfies} \qquad T' = ES$$

Special solution:

$$\mathbf{u} = \mathbf{E}^{-1}\mathbf{T} = \begin{pmatrix} \sin x \\ \cos x \end{pmatrix} \mathbf{H}$$

Solution of original problem:

$$v(x) = H(x)\sin x$$

5.5 Initial Value Problems

$u' = \delta$ has solutions $u = H + c$

Note: $u(0) = 1$ does not make sense!
However: $u(2) = 1$ is meaningful (implies $C = 0$).

General idea:

Consider $u' + Au = S$ in $D'(I)$.

Assume $J \subset I$ is open interval and $\langle S, \varphi \rangle = \langle T_f, \varphi \rangle \forall \varphi \in D(J)$ for some $f \in C^0(J)$ ("S is continuous in J").

Then any solution u is continuously differentiable in J and the initial value problem $u(\bar{x}) = \bar{u}$ for $\bar{x} = J$ has a unique solution.

Proof:

Let u be a solution of $u' + Au = S$ in $D'(I)$ then $u' + Au = f$ in $D'(J)$.
$\Longrightarrow$ $u' \in C'(J)$
Let $u^* \in C'(I)$ be the (classical) solution of $u' + Au = 0$ $u^*(\bar{x}) = \bar{u} - u(\bar{x})$
then $w = u + u^*$ satisfies $w(\bar{x}) = \bar{u}$
and $w' + Aw = S$ in $D'(J)$
and w is unique.
If $\hat{w}$ is another solution then $w - \hat{w}$ solves homogenous problem with $(w - \hat{w})(\bar{x}) = 0$, hence $w - \hat{w} = 0$.

■

Chapter 6

Convergence of Distributions

6.1 Topology in $D'(\Omega)$

Let S and $S_1, S_2, S_3, ...$ be distributions in $D'(\Omega)$. We say $S_n \to_{n\to\infty}^{D'(\Omega)} S$ or $\lim_{n\to\infty} S_n = S$ if $S_n(\varphi) \to_{n\to\infty} S(\varphi) \quad \forall\varphi \in D(\Omega)$

(or $\quad < S_n, \varphi > \to_{n\to\infty} < S, \varphi > \quad \forall\varphi \in D(\Omega) \quad$)

Example
$m_\varepsilon \to_{\varepsilon\to 0} \delta \quad in \quad D'(R^d)$

i.e. for any sequence $\varepsilon_n \to_{n\to\infty} 0$

Sloppy (but usual) notation for $T_{m_\varepsilon} \to_{\varepsilon\to 0}^{D'(R^d)} \delta$

Proof:

$T_{m_\varepsilon}(\varphi) = \int m_\varepsilon(x)\varphi(x)dx$

$= \int \frac{1}{\varepsilon^d} m(\frac{x}{\varepsilon})\varphi(x)dx \qquad y = \frac{x}{\varepsilon}, \qquad dy = \frac{1}{\varepsilon^d}dx$

$= \int m(y)\varphi(\varepsilon y)dy$

Note: $\quad m(y)\varphi(\varepsilon y) \to_{\varepsilon\to 0} \varphi(0)m(y) \qquad \forall y \in R^d \qquad$ (pointwise convergence)

and $\qquad |m(y)\varphi(\varepsilon y)| \leq ||m||_\infty ||\varphi||_\infty \mathbf{1}_{B_1(0)}(y) \qquad$ ($\varepsilon-$ independent integrable dominating function

6.2 Lebesgue dominated convergence

$T_{m_\varepsilon} \rightarrow_{\varepsilon\to 0} \int \varphi(0)m(y)dy = \varphi(0)\int m(y)dy = \varphi(0) = \delta(\varphi)$

remark:

$D'(\Omega)$ is sequentially complete, i.e. if $(S_n)_{n\in N}$ is a sequence of distributions such that $S_n(\varphi)$ converges for every $\varphi \in D$, then $S(\varphi) := lim_{n\to\infty} S_n(\varphi)$ defines a distribution.

Proof:

1) Linearity

$$\begin{aligned} S(\alpha\varphi+\beta\psi) &=^{def} lim_{n\to\infty} S_n(\alpha\varphi+\beta\psi) \\ &= lim_{n\to\infty}(\alpha S_n(\varphi)+\beta S_n(\psi)) \\ &= \alpha lim_{n\to\infty} S_n(\varphi) + \beta lim_{n\to\infty} S_n(\psi) \\ &= \alpha S(\varphi)+\beta S(\psi) \end{aligned} \tag{6.1}$$

2) Continuity

Using a contradiction ... technical

See W. Walter, Einfülerung in die Theore der Distributionen. ■

6.3 Continuity of vectorspace operations

If $(R_n)_{n\in N}$ and $(Sn)_{n\in N}$ are convergent sequences in $D'(\Omega)$ with limits $R, S \in D'(\Omega)$ and if $\alpha, \beta \in \mathbb{C}$, then $\alpha S_n + \beta R_n \rightarrow_{n\to\infty}^{D'(\Omega)} \alpha S + \beta R$

Proof:

Let $\varphi \in D(\Omega)$. Then

$< \alpha S_n + \beta R_n, \varphi >= \alpha < S_n, \varphi > +\beta < R_n, \varphi >$

$\rightarrow_{n\to\infty} \qquad \alpha < S, \varphi > +\beta < R, \varphi >=< \alpha S + \beta R, \varphi >$ ■

Example
New distributions generated by linear combinations

$(\lambda_n)_{n\in N}$ *sequence in* $\mathbb{C}$

$(\bar{x}_n)_{n\in N}$ *sequence in* Ω *with* $\{\bar{x}_n | n \in N\} \cap K$ *finite whenever* $K \subset\subset \Omega$ *(no accumulation point in* Ω *)*

Then $S_N := \sum_{n=1}^{N} \lambda_n \delta_{\bar{x}_n} \in D'(R^d)$

Even: $S := lim_{N\to\infty} S_N \in D'(R^d)$

Usual notation: $S = \sum_{n=1}^{\infty} \lambda_n \delta_{\bar{x}_n}$

e.g. "infinite row of deltas" $S = \sum_{n=-\infty}^{+\infty} \delta_n \in D'(R)$

6.4 Proof of convergence

Let $K \subset\subset \mathbb{R}^d$. Then, there exists $\bar{N}(K) \in N$

such that $\bar{x}_n \notin K \qquad \forall n \geq \bar{N}(K)$

hence: $S_n(\varphi) = S_{\bar{N}_\varphi}(\varphi) \qquad \forall n \geq \bar{N}(supp\varphi) = \bar{N}_\varphi$

so that $S_n(\varphi) \to_{n\to\infty} S_{\bar{N}\varphi}(\varphi) \qquad \forall \varphi \in D(\Omega)$

Using sequential completeness ... $S = limS_n \in D'(\Omega)$

since we did not show continuity ...

Assume $\varphi_i \to_{i\to\infty}^{D(\Omega)} 0$

show $S(\varphi_i) \to_{i\to\infty} 0$

By definition $\exists K \subset\subset R^d: \qquad supp\varphi_i \subset K \qquad \forall i \in N$

Hence $S_n(\varphi_i) = S_{\bar{N}(K)}(\varphi_i) \qquad \forall n \geq \bar{N}(K) \qquad \forall i \in N$

$\Longrightarrow \qquad S(\varphi_i) = S_{\bar{N}(K)}(\varphi_i) \to_{i\to\infty} 0$

becauce of continuity of addition and scalar multiplication,

where $S_{\bar{N}(K)}(\varphi_i) \in D'(\Omega)$ ■

6.5 Continuity of Differentiation

$(S_n)_{n\in N} \qquad D'(\Omega)$ - convergent

Then: $\nabla^\alpha lim_{n\to\infty} S_n = lim_{n\to\infty} \nabla^\alpha S_n$

Proof:

$< \nabla^\alpha S_n, \varphi >= (-1)^{|\alpha|} < S_n, \nabla^\alpha \varphi > \to_{n\to\infty} (-1)^{|\alpha|} < S, \nabla^\alpha \varphi >=< \nabla^\alpha S, \varphi > whereS = limS_n$ ■

Example

Assume $\sum_{n=1}^{\infty} S_n$ *exists in* $D'(\Omega)$

Then $\nabla^{\alpha} \sum_{n=1}^{\infty} S_n = \sum_{n=1}^{\infty} \nabla^{\alpha} S_n$

Chapter 7

Working in $\mathcal{D}'(\Omega)$

7.1 General remarks

Operations on general distributions (like differentiation, variable transformation, multiplication with functions etc.) should be extensions of the usual concepts, i.e. coincide with usual operations for regular distributions.

Remark
In other words this means that the following diagram should be commutative:

$$\begin{array}{ccc} f & \xrightarrow{\mathcal{L}} & \mathcal{L}f \\ \downarrow & & \downarrow \\ T_f & \longrightarrow & T_{\mathcal{L}f} \end{array}$$

A very simple example:

> Consider the operation "multiplication with $\lambda \in \mathbb{C}$". Clearly: $f \in \mathbb{L}^1_{\text{loc}}(\mathbb{R}^d)$ turns into $(\lambda f) \in \mathbb{L}^1_{\text{loc}}(\mathbb{R}^d)$, where $(\lambda f)(x) = \lambda f(x)$
>
> - Introduce the operation as operator: $\mathcal{L}f := \lambda f$
> - Define $\mathcal{L}$ for regular distributions in the natural way:
>
> $$\lambda T_f = \mathcal{L}T_f := T_{\mathcal{L}f} = T_{\lambda f}$$

- Manipulate $\langle T_{\lambda f}, \varphi\rangle$ into a form $\langle T_f, \mathcal{L}^*\varphi\rangle$ Here:

$$\langle T_{\lambda f}, \varphi\rangle = \langle \lambda f, \varphi\rangle = \langle f, \lambda\varphi\rangle = \langle T_f, \mathcal{L}^*\varphi\rangle$$

  with $\mathcal{L}^*\varphi := \lambda\varphi$
- Generalize $\langle \mathcal{L}T_f, \varphi\rangle = \langle T_f, \mathcal{L}^*\varphi\rangle$ to all distributions by

$$\langle \mathcal{L}S, \varphi\rangle := \langle S, \mathcal{L}^*\varphi\rangle \quad \forall\varphi \in \mathcal{D}'(\Omega)$$

- Check that this relation defines a distribution, i.e. the mapping

$$\varphi \to \langle S, \mathcal{L}^*\varphi\rangle$$

  should be <u>linear</u> and <u>continuous</u>.

If all of this is shown, then we are **done**!

To show linearity and continuity we need that the mapping

$$\mathcal{L}^* : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$$

is linear and continuous. Indeed, if this is fulfilled we get:

1. Linearity:

$$\begin{aligned} \langle S, \mathcal{L}^*(\alpha\varphi + \beta\psi)\rangle &\overset{\text{lin. of } \mathcal{L}^*}{=} \langle S, \alpha\mathcal{L}^*\varphi + \beta\mathcal{L}^*\psi\rangle \\ &\overset{\text{lin. of } S}{=} \alpha\langle S, \mathcal{L}^*\varphi\rangle + \beta\langle S, \mathcal{L}^*\psi\rangle \end{aligned}$$

2. Continuity:

$$\begin{aligned} \varphi_i \underset{i\to\infty}{\overset{\mathcal{D}(\Omega)}{\longrightarrow}} 0 &\overset{\text{cont. of } \mathcal{L}^*}{\Longrightarrow} \mathcal{L}^*\varphi_i \underset{i\to\infty}{\overset{\mathcal{D}(\Omega)}{\longrightarrow}} 0 \\ &\overset{\text{cont. of } S}{\Longrightarrow} \langle S, \mathcal{L}^*\varphi_i\rangle \underset{i\to\infty}{\longrightarrow} 0 \end{aligned}$$

Example: For our simple example $\mathcal{L}^*\varphi = \lambda\varphi$ we easily see:

- $\varphi \in \mathcal{D}(\Omega) \Rightarrow \lambda\varphi \in \mathcal{D}(\Omega)$
  $\Longrightarrow \mathcal{L}^* : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$
- $\lambda(\alpha\varphi + \beta\psi) = \alpha(\lambda\varphi) + \beta(\lambda\psi)$
  $\Longrightarrow \mathcal{L}^*$ linear

- $\varphi_i \xrightarrow{\mathcal{D}(\Omega)} 0 \Rightarrow \exists K \subset\subset \Omega : \ \mathrm{supp}(\varphi_i) \subset K \quad \forall i$

$$\Rightarrow \mathrm{supp}(\lambda\varphi_i) \subset K \quad \forall i$$

and: $\|\nabla^\alpha \varphi_i\|_\infty \longrightarrow 0 \quad \forall \alpha$

$$\Rightarrow \|\nabla^\alpha(\lambda\varphi_i)\|_\infty = |\lambda| \|\nabla^\alpha \varphi_i\|_\infty \longrightarrow 0 \quad \forall \alpha$$

$$\Longrightarrow \mathcal{L}^* \text{ continuous}$$

Thus $\mathcal{L}$ defines a distribution.

Remark
$\mathcal{L} : \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$ *is automatically continuous:*

$$\langle \mathcal{L}S_n, \varphi \rangle = \langle S_n, \mathcal{L}^*\varphi \rangle \underset{n\to\infty}{\longrightarrow} \langle S, \mathcal{L}^*\varphi \rangle = \langle \mathcal{L}S, \varphi \rangle \quad \forall \varphi \in \mathcal{D}(\Omega)$$

7.2 Differentiation

$$\mathcal{L}f := \frac{\partial}{\partial x_i} f, \qquad f \in \mathcal{C}^1(\Omega)$$

Note:

$$\langle \mathcal{L}T_f, \varphi \rangle = \left\langle \frac{\partial f}{\partial x_i}, \varphi \right\rangle = -\left\langle f, \frac{\partial \varphi}{\partial x_i} \right\rangle = \langle T_f, \mathcal{L}^*\varphi \rangle$$

with

$$\mathcal{L}^*\varphi = -\frac{\partial}{\partial x_i}\varphi$$

Define:

$$\left\langle \frac{\partial}{\partial x_i} S, \varphi \right\rangle := \left\langle S, -\frac{\partial}{\partial x_i}\varphi \right\rangle$$

An application of the above is the construction of $\frac{\partial S}{\partial x_i}$ with the "difference quotient".
The quotient

$$\frac{S(x + he_i) - S(x)}{h}$$

is well defined for $S \in \mathcal{D}'(\Omega)(\mathbb{R}^d)$, $h \neq 0$ and $i \in \{1, .., d\}$ and the following holds:

$$\lim_{h\to 0} \frac{S(x+he_i) - S(x)}{h} = \frac{\partial S}{\partial x_i}$$

Proof:

$$\begin{aligned}
& \left\langle \frac{S(x+he_i) - S(x)}{h} - \frac{\partial S}{\partial x_i}, \varphi(x) \right\rangle_x \\
= \quad & \frac{1}{h} \left(\langle S(x+he_i), \varphi(x) \rangle_x - \langle S(x), \varphi(x) \rangle_x \right) + \left\langle S(x), \frac{\partial \varphi(x)}{\partial x_i} \right\rangle_x \\
= \quad & \frac{1}{h} \left(\langle S(x), \varphi(x - he_i) \rangle_x - \langle S(x), \varphi(x) \rangle_x \right) + \left\langle S(x), \frac{\partial \varphi(x)}{\partial x_i} \right\rangle_x \\
= \quad & - \left\langle S(x), \frac{\varphi(x) - \varphi(x-he_i)}{h} - \frac{\partial \varphi(x)}{\partial x_i} \right\rangle_x \quad \longrightarrow \quad 0\,.
\end{aligned}$$

7.3 Linear transformation of variables

Let $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ invertible matrix, $b \in \mathbb{R}^d$, $f \in \mathbb{L}^1_{loc}(\mathbb{R}^d)$. We define a transformation of variables

$$x = Ay + b, \quad f(x) \rightsquigarrow f(Ay+b)$$

as an operator

$$(\mathcal{L}f)(y) := f(Ay+b)$$

Now we try to generalize this to general distributions:

$$\begin{aligned}
\langle \mathcal{L}T_f, \varphi \rangle &= \langle T_{\mathcal{L}f}, \varphi \rangle \\
&= \langle \mathcal{L}f, \varphi \rangle \\
&= \int f(Ay+b)\varphi(y)\mathrm{d}y \\
&= \int f(x) |\det A^{-1}| \varphi(A^{-1}(x-b))\mathrm{d}y \\
&= \langle f, \mathcal{L}^*\varphi \rangle \\
&= \langle T_f, \mathcal{L}^*\varphi \rangle
\end{aligned}$$

with $(\mathcal{L}^*\varphi)(x) := |\det A^{-1}|\varphi(A^{-1}(x-b))$. The natural generalization is:

$$\langle \mathcal{L}S, \varphi \rangle := \langle S, \mathcal{L}^*\varphi \rangle \qquad \text{for } S \in \mathcal{D}'(\mathbb{R}^d)$$

In other notation:

$$\Big\langle \underbrace{S(Ay+b)}_{\text{another symbol for } \mathcal{L}S}, \varphi(y)\Big\rangle_y = \Big\langle S(x), |\det A^{-1}|\varphi(A^{-1}(x-b))\Big\rangle_x$$

To show that $\mathcal{L}S \in \mathcal{D}'(\mathbb{R}^d)$ we need some properties of $\mathcal{L}^*$:

1. $\varphi \in \mathcal{D}(\mathbb{R}^d) \Rightarrow \mathcal{L}^*\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ (chain rule)
   If supp$\varphi \subset \overline{B_R(0)}$ then supp$\mathcal{L}^*\varphi \subset \overline{B_{R/\|A^{-1}\|}(b)} \subset\subset \mathbb{R}^d$
   hence $\mathcal{L}^* : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$
2. $\mathcal{L}^*$ obviously linear
3. $\varphi_i \overset{\mathcal{D}(\Omega)}{\longrightarrow} 0 \Rightarrow \exists R : \text{supp}\varphi_i \subset \overline{B_R(0)} \quad \forall i$
   $\Rightarrow \text{supp}\mathcal{L}^*\varphi_i \subset B_{R/\|A^{-1}\|}(b)$
   and $\|\nabla^\alpha \mathcal{L}^*\varphi_i\|_\infty \underset{i\to\infty}{\longrightarrow} 0 \quad \forall\alpha$ (chain rule)
   $\Rightarrow \mathcal{L}^*\varphi_i \underset{i\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} 0$
   $\Rightarrow \mathcal{L}^*$ continuous

Examples:

- Shift: $A = I, \quad b = -h;$ notation: $(\mathcal{L}T)(x) = T(x-h)$
  defined by $\langle T(x-h), \varphi(x)\rangle_x := \langle T(x), \varphi(x+h)\rangle_x$
  then $\langle \delta(x-h), \varphi(x)\rangle_x = \langle \delta(x), \varphi(x+h)\rangle_x = \varphi(h) = \langle \delta_h(x), \varphi(x)\rangle_x$
  hence $\boxed{\delta_h(x) = \delta(x-h)}$
  $T \in \mathcal{D}'(\mathbb{R})$ is "periodic with period h" if $T(x-h) = T(x)$
- Scaling: $\varepsilon > 0, \quad A = \varepsilon I, \quad b = 0;$ notation: $(\mathcal{L}T)(x) = T(\varepsilon x)$
  defined by $\langle T(\varepsilon x), \varphi(x)\rangle_x := \left\langle T(x), \frac{1}{\varepsilon}\varphi(\frac{x}{\varepsilon})\right\rangle_x$
  for example: $\langle \delta(\varepsilon x), \varphi(x)\rangle_x = \frac{1}{\varepsilon^d}\left\langle \delta(x), \varphi(\frac{x}{\varepsilon})\right\rangle_x$
  $= \frac{1}{\varepsilon^d}\varphi(0) = \left\langle \frac{1}{\varepsilon^d}\delta(x), \varphi(x)\right\rangle_x$
  hence $\boxed{\delta(\varepsilon x) = \varepsilon^{-d}\delta(x)}$

Definition 6
$T \in \mathcal{D}'(\mathbb{R}^d)$ is called a "homogeneous distribution of order $r \in \mathbb{R}$" if

$$T(\varepsilon x) = \varepsilon^r T(x) \qquad \forall \varepsilon > 0$$

($\delta \in \mathcal{D}'(\mathbb{R}^d)$ is homogeneous of order $-d$)

- Reflection: $A = -I, \quad b = 0;$ notation: $(\mathcal{L}T)(x) = T(-x)$
  defined by $\langle T(-x), \varphi(x) \rangle_x := \langle T(x), \varphi(-x) \rangle_x$
  For $d = 1$: $T \in \mathcal{D}'(\mathbb{R})$ is called
  "even" if $T(-x) = T(x)$
  "odd" if $T(-x) = -T(x)$
  For $d > 1$: $T \in \mathcal{D}'(\mathbb{R})$ is called
  "symmetric" w.r.t. 0 if $T(-x) = T(x)$

- Rotation: $A = R, \quad b = 0, \quad RR^T = I$
  defined by $\langle T(Rx), \varphi(x) \rangle_x := \left\langle T(x), \varphi(R^T x) \right\rangle_x$
  $T \in \mathcal{D}'(\Omega)$ is called "rotaionally symmetric if
  $T(Rx) = T(x) \quad \forall R \in SO(d)$, where $SO(d)$ are the $d \times d$ rotation matrices.

7.4 Multiplication

Consider $(\mathcal{L}f)(x) := g(x)f(x)$, where g is a given function. One might define the multiplication as

$$< \mathcal{L}T_f > = < T_{\mathcal{L}f}, \varphi > = < T_{fg}, \varphi > = < fg, \varphi > = \int fg\varphi \, \mathrm{d}x \qquad (7.1)$$

But this integral might not be well defined for general g. The following examples describe some problems which might occur.

- Consider $f, g \in \mathbb{L}^1_{\text{loc}}(\mathbb{R}^d)$ given by

$$f(x) = g(x) = \frac{1}{\sqrt{|x|}} \quad \Rightarrow \quad f, g \in \mathbb{L}^1_{\text{loc}}(\mathbb{R}^d),$$
$$\text{but} \quad f(x)g(x) = \frac{1}{|x|} \quad \Rightarrow \quad f \cdot g \notin \mathbb{L}^1_{\text{loc}}(\mathbb{R}^d)$$

  That means $T_{f \cdot g}$ is not defined! The problem in this example is due to a singularity. Other sources for problems might be oscillations!

- Consider $f_n \in C^\infty([0,1])$ given by $f_n(x) = \sin(nx)$. Then

$$
\begin{aligned}
< T_{f_n}, \varphi > = < f_n, \varphi > &= \int_0^1 \sin(nx)\varphi(x)\,\mathrm{d}x \\
&= -\int_0^1 \frac{\cos(nx)}{n}\varphi'(x)\,\mathrm{d}x \overset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

  The natural candidate for the product $T_{f_n} \cdot T_{f_n}$ would of course be $< T_{f_n} \cdot T_{f_n}, \varphi > := < f_n \cdot f_n, \varphi >$. However:

$$
\begin{aligned}
< f_n \cdot f_n, \varphi > &= \int_0^1 \sin^2(nx)\varphi(x)\,\mathrm{d}x \\
&= \int_0^1 \frac{1}{2}(1+\cos(2nx))\varphi(x)\,\mathrm{d}x \\
&\longrightarrow \frac{1}{2}\int_0^1 \varphi(x)\,\mathrm{d}x = < \frac{1}{2}, \varphi > = < T_{\frac{1}{2}}, \varphi >
\end{aligned}
$$

  Hence the natural approach implies

$$
\begin{aligned}
& T_{f_n} \overset{\mathcal{D}'}{\longrightarrow} 0\,, \\
\text{whereas} \quad & T_{f_n} \cdot T_{f_n} \overset{\mathcal{D}'}{\longrightarrow} T_{\frac{1}{2}} \neq 0\,.
\end{aligned}
$$

  That means the natural multiplication cannot be continuous!

Summary: Because of possible singularities and highly oscillatory behaviour of distributions there is no reasonable extension of the usual product of functions. To define the product one has to restrict the multiplication partner.

Let $g \in C^\infty(\Omega)$ and $f \in \mathbb{L}^1_{\text{loc}}(\Omega)$, then $\mathcal{L}f \in \mathbb{L}^1_{\text{loc}}(\Omega)$ and

$$
\begin{aligned}
< \mathcal{L}T_f, \varphi > &= < T_{\mathcal{L}f}, \varphi > = < gf, \varphi > \\
&= \int gf\varphi\,\mathrm{d}x = < f, g\varphi > = < T_f, \mathcal{L}\varphi >
\end{aligned}
$$

Note: $\mathcal{L} : \mathcal{D}(\Omega) \to \mathcal{D}(\Omega)$ is linear and continuous.

That means under the above restrictions we are able to define the product as

$$
< gS, \varphi > := < S, g\varphi >\,, \tag{7.2}
$$

with the following notation:

$$\mathcal{L}S = gS \quad \text{resp.} \quad (\mathcal{L}S)(x) = g(x)S(x)$$

For the multiplication the common rules are valid:

- **Product rule**
  Let $S \in \mathcal{D}'(\Omega)$ and $a \in C^\infty(\Omega)$, then

$$\frac{\partial}{\partial x_i}(aS) = \frac{\partial a}{\partial x_i} S + a \frac{\partial S}{\partial x_i}$$

Proof:

$$\begin{aligned}
< \frac{\partial}{\partial x_i}(aS), \varphi > \quad &\overset{\text{Def.}}{=} \quad - < aS, \frac{\partial}{\partial x_i}(\varphi >) \overset{\text{Def.}}{=} - < S, a\frac{\partial}{\partial x_i}(\varphi) > \\
&\overset{\text{prod.rule}}{=} \quad - < S, \frac{\partial}{\partial x_i}(a\varphi) - \varphi \frac{\partial}{\partial x_i}(a) > \\
&\overset{\text{Def.}}{=} \quad < \frac{\partial}{\partial x_i}(S), a\varphi > + < S, \frac{\partial}{\partial x_i}(a)\varphi > \\
&= \quad < a\frac{\partial S}{\partial x_i} + \frac{\partial a}{\partial x_i} S, \varphi > \qquad \forall \varphi \in \mathcal{D}'(\mathbb{R}^d)\,.
\end{aligned}$$

- **Chain rule**
  Let $S \in \mathcal{D}'(\mathbb{R}^d)$, $A \in \mathbb{R}^{d\times d}$ invertible, $b \in \mathbb{R}^d$, then

$$\frac{\partial}{\partial y_i}\left(S(Ay+b)\right) = \sum_{j=1}^{d} \frac{\partial S}{\partial x_j}(Ay+b)\frac{\partial}{\partial y_i}(Ay+b)_j = \sum_{j=1}^{d} A_{ji}\frac{\partial S}{\partial x_j}$$

Proof:

$$\begin{aligned}
< \frac{\partial}{\partial y_i}\left(S(Ay+b)\right), \varphi(x) >_y \quad &\overset{\text{Def.}}{=} \quad - < S(Ay+b), \frac{\partial \varphi}{\partial y_i}(y) >_y \\
&= \quad -|\mathrm{det}A^{-1}| < S(x), \frac{\partial \varphi}{y_i}\left(A^{-1}(x-b)\right) >_x
\end{aligned}$$

Note:

$$\begin{aligned}
\frac{\partial}{\partial x_j}\varphi\left(A^{-1}(x-b)\right) \quad &= \quad \sum_{i=1}^{d} \frac{\partial \varphi}{\partial y_i}\left(A^{-1}(x-b)\right)\frac{\partial}{\partial x_j}\left(A^{-1}(x-b)\right)_i \\
&= \quad \sum_{i=1}^{d} \frac{\partial \varphi}{\partial y_i}\left(A^{-1}(x-b)\right) A^{-1}_{ij}
\end{aligned}$$

Now multiply the equation by A_{jk} and sum over j.
Note that $\sum_j A^{-1}_{ij} A_{jk} = \delta_{ij}$, then

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j}\varphi\left(A^{-1}(x-b)\right) = \frac{\partial \varphi}{\partial y_k}\left(A^{-1}(x-b)\right) \, .$$

Hence

$$\begin{aligned} < \frac{\partial}{\partial y_i}\left(S(Ay+b)\right), \varphi(x) >_y &= |\det A^{-1}| \sum_{j=1}^{d} A_{jk} < \frac{\partial S}{\partial x_j}(x), \varphi\left(A^{-1}(x-b)\right) >_x \\ &= < \sum_{j=1}^{d} A_{jk} \frac{\partial S}{\partial x_j}(Ay+b), \varphi(y) >_y \qquad \forall \varphi \in \mathcal{D}'(\mathbb{R}^d) \, . \end{aligned}$$

Chapter 8

Local Properties of Distributions

8.1 The Support

We have to remember that Distributions do not have a value at a point. However, Distributions can be zero in an open set.
Let $f \in L^1_{loc}(\Omega)$, G $\subset \Omega$ be an open and f=0 in G (see figure 8.1)then, $< T_f, \phi >= \int_\Omega f\phi$dx=0 $\forall\phi \in$D(G).
Extend:let S$\in D'(\Omega)$,we say S=0 in an open set G$\subset \Omega$ if S=0in the

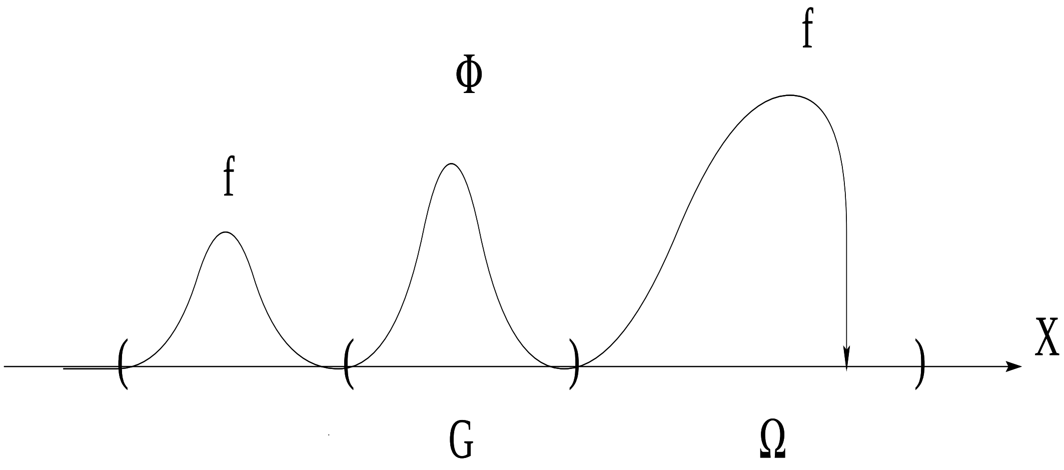


Figure 8.1:

open set G$\subset \Omega$ if (S,ϕ)=0 $\forall\phi \in$ D(G)

Note; $S_1 = S_2$ in G naturally defined as $S_1 - S_2 = 0$ in G .We can also write $S_1(x) = S_2(x), \forall x \in$ G
EXAMPLE: $\delta \in D'(R)$.
Now δ=0 in R$\setminus\{0\}$ which is same as, δ(x)=0 for x$\neq$ 0

8.1.1 Support of S$\in$D'(Ω)

Let M=$\{G \subset \Omega$/S=0 in G$\}$
Now $G_S = \bigcup_{G \in M} G$: largest open set in which S=0(which has to be shown).
Now $suppS = G_s^c \bigcap \Omega$
Example:
Now $\delta \in$ D'(R) : G$_\delta$=R$\setminus\{0\}$
$\Rightarrow$ supp$\delta = \{0\}$
Now we shall show that $S = 0$ in G_S.
Take $\phi \in D(G_S)$ then
supp$\phi \subset\subset G_S$.
That is supp$\phi \subset \bigcup_{i=1}^{p} G_i, G_i \in$ M .
Choose partition of unity with respect to $G_1, G_2, ..., G_p$.
(See Functional Analysis by Walter Rudin)
That is, $W_i \in D(G_i)$, i=1,2,3........ .p with $\sum_{i=1}^{p} W_i = 1$ $\forall x \in$supp ϕ.
Then $< S, \phi > = < S, \sum_{i=1}^{p} W_i \phi > = \sum_i < S, W_i \phi > = 0$.
(Because,$W_i \phi \in D(G_i)$ and $G_i \in M$)
Hence $S = 0$ in G_S.

8.1.2 A Natural Result:

let S$\in D'(\Omega)$,$\phi \in D(\Omega)$..
If supp S and supp ϕ are disjoint then$< S, \phi > = 0$
Proof:
SuppS $\bigcap$ Supp$\phi = \phi$
$\Rightarrow$ Supp$\phi \subset (SuppS)^c \bigcap \Omega$
$\Rightarrow \phi \in D((SuppS)^c) \bigcap \Omega)$
$\Rightarrow < S, \phi > = 0$.

8.2 Distribution with compact support

Let S $\in D'(\Omega)$ with supp S$\subset\subset \Omega$
Then $< S, \phi >$ makes sense even for $\phi \in C^{\infty}(\Omega)$.Indeed for any $\Psi \in$ D(Ω) with Ψ =1 on a neighborhood of Supp S,
$\Big\{$For construction of Ψ :
General:k$\subset\subset \Omega$ then
dist(k,$\partial\Omega$) > 0 since Ω open and k compact, say dist=ϵ
Take , k$_\epsilon = k + B_{\epsilon\backslash 2}(\theta)$
$\Rightarrow k_\epsilon$ open,k$\subset$ k$_\epsilon$.
Take 1_{k_ϵ} indicates function on K$_\epsilon$ make it smooth by going cover to
$\int_{R^d} 1_{k_\epsilon}(y) m_{\frac{\epsilon}{s}}(x-y)dy\Big\}$
We set$< S, \phi > = < S, \Psi\phi >$
(Here $\Psi\phi \in$ D(Ω))
Check:
Definition independent of choice of Ψ
Let $\hat{\Psi}$ be another choice
$\Rightarrow \frac{1}{2}(\Psi + \hat{\Psi})$ =1 on neighborhood of suppS
$\Rightarrow \hat{\Psi} - \frac{1}{2}(\Psi + \hat{\Psi}) \in D((suppS)^c \bigcap \Omega)$
$\Rightarrow < S, \hat{\Psi}\phi > = < S, \hat{\Psi}\phi > - < S, (\hat{\Psi} - \frac{1}{2}(\Psi + \hat{\Psi}))\phi >$
$= < S, \frac{1}{2}(\Psi + \hat{\Psi})\phi >$
$= < S, \Psi\phi > - < S, (\Psi - \frac{1}{2}(\Psi + \hat{\Psi}))\phi >$
$= < S, \Psi\phi >$
Therefore compact supported distributions allow a larger class of test functions.
Let$\epsilon(\Omega)$=$C^{\infty}(\Omega)$ with the following notation of convergence:
$\phi_n \to$0 in $\epsilon(\Omega)$ as n$\to \infty$
$\Rightarrow \| \bigtriangledown^{\alpha}\phi_n \|_{\infty,k} \to 0$ as $n \to \infty, \forall\alpha \in N_0^n$ and $\forall k \subset\subset \Omega$
Note: D(Ω) $\subset \epsilon(\Omega)$: $\phi_n \to$0 in D(Ω)
$\Rightarrow \phi_n \to 0$ in $\epsilon(\Omega)$
Let S$\in$D'(Ω) , supp S$\subset\subset \Omega, \phi_n \to 0$ in$\epsilon(\Omega)$ as n $\to \infty$ then
$< S, \phi_n > \to 0$ as $n \to \infty$
That is S is a continuous linear functional on $\epsilon(\Omega)$
proof:
$< S, \phi_n > = < S, \Psi\phi_n >$
Now $\Psi\phi_n \in D(\Omega)$ and $\Psi\phi_n \to$0 in $D(\Omega)$ as n $\to \infty$)
(beacuse supp $\Psi\phi_n \subset supp\Psi, \forall$ n and
$\| \bigtriangledown^{\alpha}(\Psi\phi_n) \|_{\infty} \to 0$ as $n \to \infty, \forall\alpha$)
$\Rightarrow < S, \Psi\phi_n > \to 0$ as $n \to \infty$
Conversely if S is a continuous linear functional on $\epsilon(\Omega)$ then S$\in D'(\Omega)$.

and $suppS \subset\subset \Omega$)
Hence:
$\epsilon'(\omega)$=set of compact supported distributions.
Proof:
Let $K_n = \{x \in \Omega / dist(x, \partial\Omega) \geq \frac{1}{n}$ and $\mid x \mid \leq n\}$
Now ,$K_n$is compact $\bigcup_{n=1}^{\infty} k_n = \Omega$.
We shall show that for S$\in \epsilon'(\Omega)$ there exists n$\in$ N and c$>$ 0 such that
$|< S, \phi >| \leq c \max_{|\alpha| \leq n} \parallel \nabla^k \phi \parallel_{\infty, k_n}$
We shall prove this by contradiction,
That is,for all n $\exists \phi_n \in \epsilon(\Omega) : |< S, \phi_n >| \geq n \max_{|\alpha| \leq n} \parallel \nabla^\alpha \phi \parallel_{\infty, k_n}$
Set $\Psi_n = \frac{\phi_n}{n \max_{|\alpha| \leq n} \| \nabla^\alpha \phi \|_{\infty, k_n}}$.
Now $\Psi_n \in \epsilon(\Omega)$.
Then, $|< S, \Psi_n >| \geq 1$ and $\Psi_n \to 0$ in $\epsilon(\Omega)$ as $n \to \infty$ which contradicts continuity of S.
$\Big\{$Proof of $\Psi_n \to 0$ as n $\to \infty$:
Because for any $\alpha \in N_0^d, K \subset\subset \Omega$,there exists $n_0 \in$ N : $\mid \alpha \mid < n_0, K \subset K_{n_0}$
$\parallel \nabla^\alpha \phi_n \parallel_{\infty, k} = \frac{1}{n} \underbrace{\frac{\parallel \nabla^\alpha \phi \parallel_{\infty, k_n}}{\max_{|\alpha| \leq n} \parallel \nabla^\alpha \phi \parallel_{\infty, k_n}}}_{\leq 1} \Big\}$
Obviously,$|< S, \phi >| \leq C \max_{|\alpha| \leq n} \parallel \nabla^\alpha \phi \parallel_{\infty, k_n}$
implies supp S$\subset K_n$.
Indeed ,for $\phi \in D(K_n^c \bigcap \Omega)$,
$|< S, \phi >| \leq C \max_{|\alpha| \leq n} \parallel \nabla^\alpha f \parallel_{\infty, k_n} = 0$.
Summary:
$\epsilon(\Omega) \supset D(\Omega)$
$\epsilon'(\Omega) \subset D'(\Omega)$.
Even $\epsilon'(\Omega)$ dense in $D'(\Omega)$.
Trick:Cut off
LetK_n the sequence as before.
Let $\chi_n \in D(\Omega), \chi_n = 1$ on K_n.
Let $S \in D'(\Omega)$
Consider $S_n = \chi_n S$
Note:
supp $S_n = supp \chi_n \subset\subset \Omega$
because for any$\phi \in D((supp \chi_n)^c \bigcap \Omega)$,
We have$< S_n, \phi >=< \chi_n S, \phi >=< S, \chi_n \phi >= 0$
and since $\chi_n \phi \to \phi$ in $D(\Omega)$ as n $\to \infty, \forall \phi \in D(\Omega)$.
We have$< S_n, \phi >=< S, \chi_n \phi >\to< S, \phi >$ as n $\to \infty$.
That is $S_n \to S$ in $D'(\Omega)$ as n $\to \infty$

8.3 Order of Distribution:

An estimate:
Let S$\in D'(\Omega), K \subset\subset \Omega$
then there exists n$\in$N and C>0
$|< S, \phi >| \leq C \max_{|\alpha| \leq n} \| \nabla^{\alpha}\phi \|_{\infty}$
$\forall \phi \in D(\Omega)$ with $supp\phi \subset K$
Proof;
Let ,$\psi \in D'(\Omega), \Psi = 1$ on neighborhood of K.
$\Rightarrow \Psi\phi = \phi$.
Note:
$< S, \phi > = < S, \Psi\phi > = < \Psi S, \phi >$
Use previous estimate,.... $\exists C > 0, n_0 \in N$
$|< S, \phi >| = |< \Psi S, \phi >| \leq C \max_{|\alpha| \leq n_0} \| \nabla^{\alpha}\phi \|_{\infty, k_{n_0}} \leq C \max_{|\alpha| \leq n_0} \| \nabla^{\alpha}\phi \|_{\infty}$
Definition:
$S \in D'(\Omega)$ is called a distribution of order n if for any $K \subset\subset \Omega$ there exists C > 0 such that
$|< S, \phi >| \leq C \max_{|\alpha| \leq n} \| \nabla^{\alpha}\phi \|_{\infty}, \forall \phi \in D(\Omega), supp\phi \subset K.$
Note:
1.Any$S \in \epsilon'(\Omega)$ is of finite order
2.Obivously $\nabla^{\alpha}\delta$ is of order$| \alpha |$,
since,
$|< \nabla^{\alpha}\delta, \phi >| = | \nabla^{\alpha}\phi(0) | \leq \max_{|\beta| \leq \alpha|} \| \nabla^{\beta}\phi \|_{\infty}$.

Chapter 9

The direct product

Throughout this section, we set

$$X = \mathbb{R}^m, Y = \mathbb{R}^n, Z = X \times Y = \mathbb{R}^{m+n}$$

Definition 7
Let $f \in \mathbb{L}^1_{loc}(X)$, $g \in \mathbb{L}^1_{loc}(Y)$ *, then the direct product is* $h(x,y) := f(x)g(y)$*, often abbreviated by* $h := f \otimes g$.

Note : $h \in \mathbb{L}^1_{loc}(Z)$ and for $\varphi \in D(Z)$:

$$\langle h(x,y), \varphi(x,y)\rangle_{(x,y)} = \langle f(x)g(x), \varphi(x,y)\rangle_{(x,y)}$$
$$= \int_Z h(x,y)\varphi(x,y)\,d(x,y) = \int_Z f(x)g(y)\varphi(x,y)\,d(x,y)$$
$$= \int_X f(x)\int_Y g(y)\varphi(x,y)\,dy\,dx = \langle f(x), \langle g(y), \varphi(x,y)\rangle_Y\rangle_X$$

9.1 Extension to distributions

Extend the concept to distributions to allow $\delta \otimes \delta$, ...

Lemma 2
Let $G \in D'(Y)$, $\varphi \in D(Z)$*. Then*

$$\begin{array}{rcl} \mathcal{L} : D(Z) & \longrightarrow & D(X) \\ \varphi & \longrightarrow & \langle G(y), \varphi(.,y)\rangle_Y \end{array}$$

is a continuous mapping.

Ideas for the proof:

- $\mathcal{L}$ is obviously linear.
- $\exists R : \varphi \in D(Z) \to (\mathcal{L}\varphi)(x) = 0$ for $|x| > R$

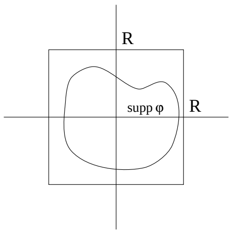


Figure 9.1:

- $\frac{1}{h}\left(\mathcal{L}\varphi\right)(x+he_i) - (\mathcal{L}\varphi)(x))$

$$= \left\langle G(y), \frac{1}{h}\left(\varphi(x+he_i,y) - \varphi(x,y)\right)\right\rangle_Y \xrightarrow[h\to 0]{} \left\langle G(y), \frac{\partial}{\partial x_i}\varphi(x,y)\right\rangle_Y$$

$$\text{since } \frac{1}{h}\left(\varphi(x+he_i,y) - \varphi(x,y)\right) \xrightarrow[h\to 0]{D(Z)} \frac{\partial}{\partial x_i}\varphi(x,y)$$

- Using the previous point and induction, we can show, that $\mathcal{L}\varphi \in C^\infty(X)$; $\nabla^\alpha\mathcal{L}\varphi = \mathcal{L}\nabla_x^\alpha\varphi$. Together with the second point, we get, that $\mathcal{L}\varphi \in D(X)$.
- Continuity of $\mathcal{L} : D(Z) \longrightarrow D(X)$:

$$\varphi_k \xrightarrow[k\to\infty]{D(Z)} 0 \Rightarrow supp\,\varphi_k \subset K \subset\subset \Omega \;\forall k \in \mathbb{N}$$

$$\Rightarrow \text{ in the second point } R \text{ can be chosen uniformly}$$

$$\Rightarrow supp\,\mathcal{L}\varphi_k \subset\subset B_R(0) \;\forall k \in \mathbb{N}.$$

$$\begin{aligned} \left|\left(\nabla^\beta\mathcal{L}\varphi_k\right)(x)\right| &= \left|\mathcal{L}\left(\nabla_x^\beta\varphi_k\right)(x)\right| = \\ &= \left|\left\langle G(y), \nabla_x^\beta\varphi_k(x,y)\right\rangle_Y\right| \\ &\overset{\text{Estimate by section 8.3}}{\leq} C\max_{|y|\leq n_0}\left\|\nabla_y^\gamma\nabla_x^\beta\varphi_k(x,.)\right\|_\infty \xrightarrow[k\to\infty]{} 0 \end{aligned}$$

uniformely in X, because

$$\|\nabla^\alpha\varphi_k\| \xrightarrow[k\to\infty]{} 0 \;\forall\alpha$$

Definition 8
Direct product of distributions:
Let $F(x) \in D'(X)$, $G(y) \in D'(Y)$. Then the direct product

$$H(z) := F(x) \otimes G(y)$$

is defined by

$$\langle H(z), \varphi(z) \rangle_Z := \langle F(x), \langle G(y), \varphi(x,y) \rangle_Y \rangle_X \,.$$

Note: r.h.s. really defines a distribution with previous result.

Lemma 3
Formula for support of the direct product:

$$supp\, H = supp\, F \times supp\, G$$

Proof:
$x_0 \in supp\, F$ and $y_0 \in supp\, G$ implies that $\exists \psi_\varepsilon \in D\left(B_\varepsilon(x_0)\right)$, $\eta_\varepsilon \in (B_\varepsilon(y_0))$ with $\langle F, \psi_\varepsilon \rangle \neq 0$, $\langle G, \eta_\varepsilon \rangle \neq 0$. Hence:

$$\begin{aligned} \langle H, \psi_\varepsilon \otimes \eta_\varepsilon \rangle &= \langle F(x), \langle G(y), \psi_\varepsilon(x) \eta_\varepsilon(y) \rangle \rangle \\ &= \langle F, \psi_\varepsilon \rangle \langle G, \eta_\varepsilon \rangle \neq 0 \end{aligned}$$

$\Rightarrow H$ does not vanish in any neighbourhood of $z_0 = (x_0, y_0)$
$\Rightarrow z_0 \in supp\, H \Rightarrow supp\, F \times supp\, G \subset supp\, H$.

Conversely:
Let $z_0 = (x_0, y_0) \in supp\, H$.

- $y_0 \notin supp\, G \Rightarrow \exists \varepsilon > 0 : \ \forall \eta \in D\left(B_\varepsilon(y_0)\right) : \ \langle G, \eta \rangle = 0$
  $\Rightarrow \forall \varphi \in D\left(B_\varepsilon(x_0) \times B_\varepsilon(y_0)\right) \ \langle H, \varphi \rangle = 0$ - contradiction.
- Assume now, $x_0 \notin supp\, F$.
  Then $\exists \varepsilon > 0 : \ \forall \psi \in D\left(B_\varepsilon(x_0)\right) \ \langle F, \psi \rangle = 0$.
  Note: $\varphi \in D\left(B_\varepsilon(x_0) \times B_\varepsilon(y_0)\right)$
  $\Rightarrow supp\, \mathcal{L}\varphi \subset B_\varepsilon(x_0) \Rightarrow \langle H, \varphi \rangle = 0$ - contradiction.

9.2 Examples

Example

$$\delta(x) \otimes \delta(y) = \delta(z)$$

Indeed,

$$\begin{aligned}&\langle \delta(x)\otimes\delta(y),\varphi(x,y)\rangle_{(x,y)}\\ &=\left\langle \delta(x),\,\langle\delta(y),\varphi(x,y)\rangle_y\right\rangle_x\\ =\langle\delta(x),\varphi(x,0)\rangle=\varphi(0,0)=&\langle\delta(z),\varphi(z)\rangle_z\end{aligned}$$

Example
$g\in\mathbb{L}^1_{loc}(y)$

$$\langle\delta(x)\otimes g(y),\varphi(x,y)\rangle=\int_y g(y)\varphi(0,y)\,dy$$

e.g. $g(y)=1$ *(call this function* $y\longrightarrow 1(y)$*), then*

$$\langle\delta(x)\otimes 1(y),\varphi(x,y)\rangle=\int_y \varphi(0,y)\,dy$$

9.3 Properties

1. Every $\varphi\in D(z)$ can be approximated in $D(z)$ by finite linear combinations of the form
$$\varphi(z)=\sum\psi_i(x)\,\eta_i(y),\text{ where }\psi_i\in D(x),\ \eta_i\in D(y)$$
Proof: Use approximation by polynomials (Weierstrass), which are exactly of that form, e.g. with $x\in\mathbb{R}$, $y\in\mathbb{R}$:
$$P(x,y)=\alpha_0+\alpha_1x+\alpha_2y+\alpha_{12}xy+\alpha_{11}x^2+\alpha_{22}y^2+...$$

2. As immediate consequence: $H\in D'(z)$ is uniquely defined by its action on test functions with product structure.
Proof: Let $H_1,H_2\in D'(z)$ with
$$\langle H_1,\,\psi\otimes\eta\rangle=\langle H_2,\,\psi\otimes\eta\rangle\,\forall\psi\in D(x),\eta\in D(y)$$
Show: $H_1-H_2=0$.
Let $\varphi\in D(z)\Rightarrow\exists\,(\gamma_k)_{k\in\mathbb{N}}$:
$$\gamma_k=\sum_{i=1}^{N_k}\psi_i^{(k)}\otimes\eta_i^{(k)}\text{ with }\gamma_k\xrightarrow{D(z)}\varphi$$
then:
$$\langle H_1-H_2,\varphi\rangle=\lim_{n\to\infty}\langle H_1-H_2,\gamma_k\rangle=\lim_{n\to\infty}\sum_{i=1}^{N_k}\left\langle H_1-H_2,\psi_i^{(k)}\otimes\eta_i^{(k)}\right\rangle=0.$$

3. **Commutativity** (Fubini's theorem):

$$\langle F \otimes G, \varphi \rangle = \Big\langle F(x), \langle G(y), \varphi(x,y) \rangle_y \Big\rangle_x \tag{9.1}$$

$$= \langle G(y), \langle F(x), \varphi(x,y) \rangle_x \rangle_y \tag{9.2}$$

Proof: (9.1) and (9.2) coincide for $\varphi = \psi \otimes \eta$.

4. **Associativity:**

$$(F \otimes G) \otimes H = F \otimes (G \otimes H)$$

Proof: The same trick.

5. **Continuity:**

$$F_k \xrightarrow{D'(x)} F \ \Rightarrow \ F_k \otimes G \xrightarrow{D'(z)} F \otimes G$$

Proof:

$$\langle F_k \otimes G, \varphi \rangle = \langle F_k, \mathcal{L}\varphi \rangle \ \longrightarrow \ \langle F, \mathcal{L}\varphi \rangle = \langle F \otimes G, \varphi \rangle$$

6. **Derivatives:**

$$\nabla_x^\alpha \nabla_y^\beta \, F(x) \otimes G(y) = (\nabla_x^\alpha \, F) \otimes (\nabla_y^\beta \, G)$$

Proof: Use test functions $\varphi = \psi \otimes \eta$

$$\begin{aligned} &\Big\langle \nabla_x^\alpha \nabla_y^\beta \, F(x) \otimes G(y), \psi(x)\,\eta(y) \Big\rangle \\ &= (-1)^{|\alpha|+|\beta|} \Big\langle F(x) \otimes G(y),\, (\nabla_x^\alpha \psi)(x)\,(\nabla_y^\beta \eta)(y) \Big\rangle \\ &= (-1)^{|\alpha|} \langle F, \nabla^\alpha \psi \rangle \ (-1)^{|\beta|} \Big\langle G, \nabla^\beta \eta \Big\rangle \\ &= \langle \nabla^\alpha F,\, \psi \rangle \ \Big\langle \nabla^\beta G, \eta \Big\rangle = \Big\langle (\nabla^\alpha F) \otimes (\nabla^\beta G),\, \psi \otimes \eta \Big\rangle \end{aligned}$$

Chapter 10

Convolutions

10.1 Definition of Convolution on Distribution

Let

$$f \in \mathbb{L}^p(\mathbb{R}^d), \quad g \in \mathbb{L}^q(\mathbb{R}^d), \qquad \frac{1}{p} + \frac{1}{q} = 1$$

or

$$f \in \mathbb{L}^p_{loc}(\mathbb{R}^d), \quad g \in \mathbb{L}^q_0(\mathbb{R}^d),$$

g with compact support, then the convolution product

$(f * g)(x) := \int_{\mathbb{R}^d} f(y)g(x-y)dy$
is well defined. $(Hölder: \quad y \to f(y)g(x-y) \in \mathbb{L}^1(\mathbb{R}^d) \ \ \forall x,$

$\quad |f * g| \leq \|f\|_{\mathbb{L}^p} \|g\|_{\mathbb{L}^q}).$

<u>Generalization</u> to distributions:

$< f * g, \varphi >= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(x-y)dy\varphi(x)dx$

$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)g(z)\varphi(y+z)dydz \qquad (byputtingx - y = z, dx = dz)$

$=< f \otimes g(y,z), \varphi(y+z) >_{(y,z)}$

which suggests to define

$< T * S, \varphi >:=< T \otimes S(y,z), \varphi(y+z) >_{(y,z)}$

Attention!

$(y, z) \longrightarrow \varphi(y+z) \notin \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^d)$, unless $\varphi \equiv 0$.

note:

$$\varphi(y+z) \neq 0 \Longleftrightarrow y+z \in supp\varphi \Longleftrightarrow z \in \bigcup_{r \in supp\varphi} \{r-y \mid y \in \mathbb{R}^d\}$$

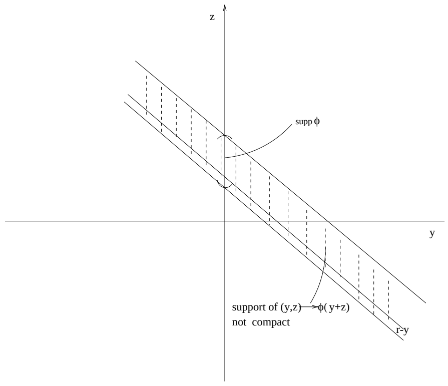


Figure 10.1:

However, $< T \otimes S(y, z), \varphi(y+z) >$ makes sense in special cases:

1) suppT or supp S is compact.

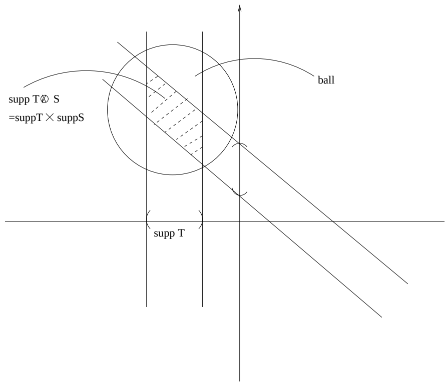


Figure 10.2: special case 1

take $\chi \in \mathbb{D}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi \equiv 1$ in this ball, we can get

$< T \otimes S(y,z), \varphi(y,z) >=< T \otimes S(y,z), \chi(x,y)\varphi(y,z) >$

the right hand side is well defined, since $\chi(x,y)\varphi(y,z)$ is a test function.

2)

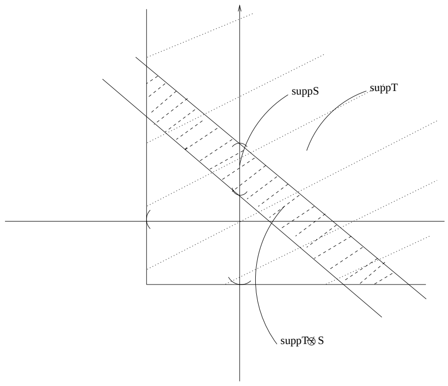


Figure 10.3: special case 2

the case 2 can be worked out as case 1.

Let $T, S \in \mathbb{D}^{'}(\mathbb{R}^d)$ with either T or S having compact support. then, the convolution $T * S \in \mathbb{D}^{'}(\mathbb{R}^d)$ is defined by

$< T * S, \varphi >:=< T(x)S(y), \chi(x,y)\varphi(x+y) >_{(x,y)}$

where $\chi = \mathbf{1}^{\varepsilon}_{suppT \times suppS}$ is a smooth function which is supported in a neibourhood of $suppT \times suppS$.

Remarks

- well defined, since $\chi(x,y)\varphi(x+y)$ defines test function in $\mathbb{D}(\mathbb{R}^d \times \mathbb{R}^d)$.
- Defination independent of χ

take any other $\widetilde{\chi}$, which is one on neighbourhood of $suppT \times suppS$ and zero in some distance, take some functional.

- obviously linear in φ
- obviously continuous in φ

because $\varphi_k \longrightarrow 0 \Longrightarrow \exists K \subset\subset \mathbb{R}^d \quad supp\varphi_k \subset K$

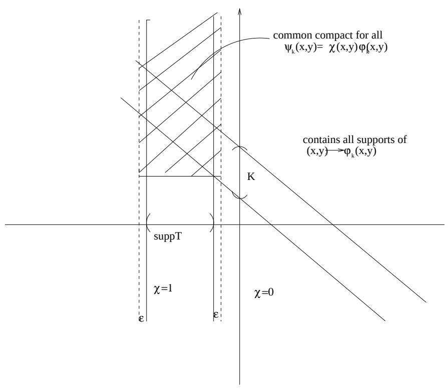


Figure 10.4:

note:

$$\bigtriangledown^{\alpha}\psi_k = \sum_{0\le\beta\le\alpha} C_{\alpha\beta} \bigtriangledown^{\beta} \chi(x,y) \bigtriangledown^{\alpha-\beta} (\varphi_k(x+y))$$

derivation of φ_k goes to zero uniformly. so we can get

$\| \bigtriangledown^{\alpha} \psi_k\|_{\infty} \longrightarrow 0 \qquad \forall\alpha$

use continuity of tensor product.

10.2 Properties of Convolution

- $suppT * S \subset suppT + suppS$

Assume $C \notin suppT + suppS$,

$\Longleftrightarrow C \neq x+y \qquad \forall x \in suppT, \quad y \in suppS$

$\Longleftrightarrow \{(x,y)|x+y=c\}$ has no intersection with $suppT \times suppS$

$\Longrightarrow < T * S, \varphi >= 0 \qquad \forall\varphi \in \mathbb{D}(B_{\delta}(C))$

$\Longrightarrow C \notin suppT * S$

- $T * \delta = T$

$< T * \delta, \varphi >=< T(x)\delta(y), \varphi(x+y) >$

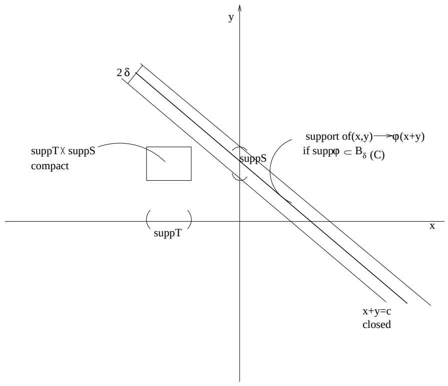


Figure 10.5:

$=< T(x), < \delta(y), \varphi(x+y) >_y>_x$

$=< T(x), \varphi(x) >_x=< T, \varphi >$

- $\bigtriangledown^\alpha T = T * (\bigtriangledown^\alpha \delta)$

$< T * \bigtriangledown^\alpha \delta, \varphi >=< T(x), (-1)^{|\alpha|} < \delta(y), \bigtriangledown_y^\alpha \varphi(x+y) >_y>_x$

$= (-1)^{|\alpha|} < T(x), \bigtriangledown^\alpha \varphi(x) >=< \bigtriangledown^\alpha T, \varphi >$

- $S * T = T * S$

$< S * T, \varphi >=< S(x)T(y), \mathbf{1}^\varepsilon_{suppS \times suppT}(x,y)\varphi(x+y) >$

$=< S(x)T(y), \mathbf{1}^\varepsilon_{suppT \times suppS}(y,x)\varphi(y+x) >$

Apply linear transformation: $A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$ $\quad , A = A^{-1}$

$=< A(S \otimes T)(x,y), \nVdash^\varepsilon_{suppT \times suppS}(x,y)\varphi(x+y) >$

$=< T(x)S(y), \nVdash^\varepsilon_{suppT \times suppS}(x,y)\varphi(x+y) >$

$=< T * S, \varphi >$

because:

$< A(S \otimes T), \varphi \otimes \psi >$

$=< S(x)T(y), \varphi \otimes \psi(A \begin{pmatrix} x \\ y \end{pmatrix}) >$

$=< S(x)T(y), \varphi(y)\psi(x) >$

$=< S, \psi >< T, \varphi >$

$=< T(x)S(y), \varphi(x)\psi(y) >$

$=< T \otimes S, \varphi \otimes \psi >$

then get $A(S \otimes T) = T \otimes S$

- $(\alpha S + \beta T) * U = \alpha S * U + \beta T * U$

if e.g U has compact support, or S and T have compact support.

• $(S * T) * U = S * (T * U)$

Attention: two of S, T, U have to be compactly supported.

a). $(S,T) \rightsquigarrow S * T$ with compact support $\rightsquigarrow (S * T) * U$ well defined.

b). $(S,U) or (T,U) \rightsquigarrow S * T \in \mathbb{D}' \rightsquigarrow (S * T) * U$ well defined.

$< (S * T) * U, \varphi >=< (S * T)(x)U(y), \varphi(x + y) >$

$=< U(y). < (S * T)(x), \varphi(x + y) >_x>_y$

$=< U(y), < S(a)T(b), \varphi(a + b + y) >_{(a,b)}>_y$

$=< S(a)T(b)U(y), \varphi(a + b + y) >$

$=< S(a), < T(b)U(y), \varphi(b + y + a) >_{(b,y)}>_a$

$=< S(a), < (T * U)(x), \varphi(x + a) >_{(x)}>_a$

$=< S * (T * U), \varphi >$

• $\bigtriangledown^\alpha (S * T) = (\bigtriangledown^\alpha S) * T = S * (\bigtriangledown^\alpha T)$

since

$$\bigtriangledown^\alpha (S * T) = \bigtriangledown^\alpha \delta * (S * T) = (\bigtriangledown^\alpha \delta * S) * T = (\bigtriangledown^\alpha S) * T$$

and $\bigtriangledown^\alpha (S * T) = (S * T) * \bigtriangledown^\alpha \delta = S * (\bigtriangledown^\alpha T)$

• Continuity

$(S_n)_{n\in\mathbb{N}} \subset \mathbb{D}^{'}(\mathbb{R}^d)$, $T \in \mathbb{D}^{'}(\mathbb{R}^d)$ with either suppT compact

or $suppS_n \subset K \subset\subset \mathbb{R}^d \qquad \forall n \in \mathbb{N}$

then $S_n * T \in D^{'}$

and $S_n * T \longrightarrow S * T$ while $n \to \infty$

proof:

$< S_n * T, \varphi >=< S_n(x) \otimes T(y), \chi(x,y)\varphi(x+y) >$

$\longrightarrow < S(x) \otimes T(y), \chi(x,y)\varphi(x+y) >=< S * T, \varphi >$

(using continuity of $\otimes$)

• Convolution with a test function $\psi \in \mathbb{D}^{'}(\mathbb{R}^d)$

$x \longrightarrow (S * \psi)(x) =< S(y), \psi(x-y) >_y \in C^{\infty}(\mathbb{R}d)$

$< S * \psi, \varphi >$

$=< S(x), < \psi(y), \varphi(x+y) >_y >_x \qquad (-z = x + y)$

$=< S(x), < \varphi(-z), \psi(-z-x) >_z >_x \qquad (\breve{f} := f(-x))$

$=< S * \breve{\varphi}, \breve{\psi} >$

$=< \breve{\varphi} * S, \breve{\psi} >$

$=< \breve{\varphi}(x), < S(y), \breve{\psi}(x+y) >_y >_x$

$=<< S(y), \breve{\psi}(-x+y) >_y, \varphi(x) >_x$ then, we get, $S*\psi(x) =< s(y), \psi(-(-x+y)) >_y$.

use for regularization, $\mathbb{D}$ dense in $\mathbb{D}^{'}$

Show $\mathbb{D}(\Omega)$ in $\varepsilon^{'}(\Omega)$, we have known already $\varepsilon^{'}(\Omega)$ dense in $\mathbb{D}^{'}(\Omega)$,

$S \in \varepsilon^{'}(\Omega) \Longleftrightarrow S \in \mathbb{D}^{'}(\Omega)$ and $suppS \subset\subset \Omega$

$\Longrightarrow \exists \varepsilon_0 > 0$ such that $suppS + B_{\varepsilon_0}(0) \subset \Omega \qquad \forall 0 \leq \varepsilon \leq \varepsilon_0$

then $S_\varepsilon = S * M_\varepsilon$, see chapter 1, $suppM_\varepsilon = \overline{B_\varepsilon(0)}$

$suppS_\varepsilon \subset suppS + suppM_\varepsilon \subset\subset \Omega$ for $\varepsilon < \varepsilon_0$

and $S_\varepsilon \in C^{\infty}(\mathbb{R}) \Longrightarrow S_\varepsilon \in \mathbb{D}(\Omega)$

we know

$$M_\varepsilon \longrightarrow \delta \quad while \quad \varepsilon \to 0 \qquad (chapter \ \ 6)$$

hence,

$$S * M_\varepsilon \longrightarrow S * \delta = S \quad while \quad \varepsilon \to 0$$

S is approximated by $S_\varepsilon \in \mathbb{D}(\Omega)$, also while $\varepsilon \longrightarrow 0$

$\bigtriangledown^\alpha(S_\varepsilon) = \bigtriangledown^\alpha(S * M_\varepsilon) = S * (\bigtriangledown^\alpha M_\varepsilon) = (\bigtriangledown^\alpha S) * M_\varepsilon$

$\longrightarrow \bigtriangledown^\alpha S * \delta = \bigtriangledown^\alpha S$

so, we can get that every $S \in \mathbb{D}^{'}(\Omega)$ can be approximated by test functions.

Chapter 11

Application to PDE's

Linear partial differential operator with constant coefficients
Let $u \in D'(\Re^d)$

$$LU := \sum_{|\alpha| \leq m} a_\alpha \nabla^\alpha u$$

where m is order of the operator L if $a_\alpha \neq 0 \; for \; some \; \alpha \; with |\alpha| = m$

Theorem of Malgrange and Ehrenpreis

There exists a solution $u \in D'(\Re^d)$ to the problem $Lu = \delta$.
Any such u is called fundamental solution (For the proof see : W. Walter or Walter Rudin - Functional Analysis)

More general problem : FInd $v \in D'(\Re^d)$ such that
(*) $Lv = f$, where $f \in \varepsilon'(\Re^d)$

Result : If u is a fundamental solution then $v = u * f$ solves (*)
$(u \in D'(\Re^d) \; and \; f \in \varepsilon'(\Re^d) \Longrightarrow v \in D'(\Re^d))$

Proof : $Lv = L(u * f) = (Lu) * f = \delta * f = f \quad \boxplus$

Example : Laplace operator

$$a_\alpha = \begin{cases} 1 \; if \; \alpha = 2e_i & i \in \{1, 2, 3, ..., d\} \\ 0 \;\; else \end{cases}$$

$$L = \sum a_\alpha \nabla^\alpha = \sum_{i=1}^{d} \nabla^{2e_i} = \sum_{i=1}^{d} \frac{\eth^2}{\eth x_i^2} = \frac{\eth^2}{\eth x_1^2} + ... + \frac{\eth^2}{\eth x_d^2} = \nabla$$

for d=3 , a fundamental solution is $u(x) = -\frac{1}{4\pi|x|}$
Note : $u \in L^1_{loc}(\Re^3)$

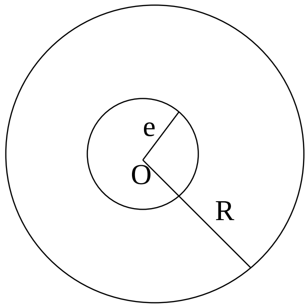
Proof : Calculate Δu
a) $for\ x \neq 0 \quad ....\Delta\ u = 0 \qquad (classically)$
b) $for\ x = 0 \qquad \Delta u = ??.... \quad (distributional\ derivative)$

$$< \Delta u, \varphi > \underbrace{=}_{\text{by } Def} < u, \Delta\varphi > = \int_{(\Re^3)} \Delta\varphi(x)dx, \qquad \varphi\ is\ a\ test\ function$$

$$\underbrace{=}_{\text{like Cauchy Principal Value}} \lim_{\epsilon \to 0} \left(\int_{|x| \le \epsilon} u\Delta\varphi dx + \int_{\epsilon < |x|} u\Delta\varphi dx \right)$$

$$1) \int_{|x| \le \epsilon} u\Delta\varphi dx \to 0 \qquad since \quad u \in L^1_{loc}$$

$$2) \int_{\epsilon < |x|} u\Delta\varphi dx = \int_{\epsilon < |x| < R} u\Delta\varphi dx \quad , if supp\varphi \subset B_R(0)$$



Note : $u\Delta\varphi = u\ div(\nabla\varphi) = div(u\nabla\varphi) - \nabla u \nabla\varphi$

$$x \neq 0 : \quad 0 = (\Delta u)\varphi = \varphi div(\nabla u) = div(\varphi\nabla u) - \nabla u \Delta\varphi$$

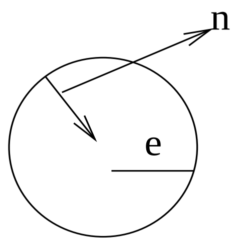
Subtract :

$$u\Delta\varphi = div(u\nabla\varphi - \varphi\nabla u), \quad x \neq 0$$

Applying Gauss theorem and using $\varphi|_{|x|=R} = 0$ *yields*

$$\int_{\epsilon<|x|} u\Delta\varphi dx = \int_{\underbrace{|x|=\epsilon}_{vol=4\pi\epsilon^2}} u\frac{\eth\varphi}{\eth n} - \varphi\frac{\eth u}{\eth n}\ d\sigma \longrightarrow_{\epsilon\to 0} 0 \quad \varphi(0)$$

Note : $\frac{\eth u}{\eth n} = -\frac{\eth}{\eth r}(\frac{1}{4\pi r})$ n - outward normal



Hence :

$$< \Delta u, \varphi >= \varphi(0) =< \delta, \varphi > \Longrightarrow \Delta u = \delta \quad \boxplus$$

More general : Poison equation

$$\Delta\Phi = \rho$$

$$\rho \textit{ mass density} \to \Phi \textit{ gravitational potential}$$

$$\rho \textit{ charge density} \to \Phi \textit{ electrostatic potential}$$

Solution : $\Phi = u * \rho$
Assume $\rho \in L_0^\infty(R^3)$ bounded + compact support
since $u \in L^1_{loc} \Longrightarrow u * \rho \in L^1_{loc}(\Re^3)$ and $\Phi(x) = \frac{-1}{4\pi}\int_{\Re^3}\frac{\rho(y)}{|x-y|}dy$

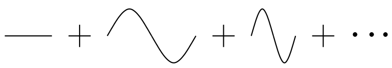
Chapter 12

Fourier Series

12.1 Basic Idea

Decompose periodic signal $f(t)$

into standard periodic signal $\sin, \cos$

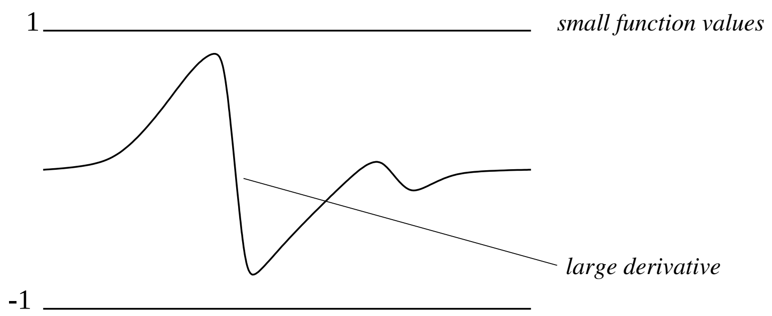


with specific amplitudes (a_k and b_k) and frequencies (kt):

$$f(t) = a_0 + \sum_{k=1}^{\infty}(a_k cos(kt) + b_k sin(kt))$$

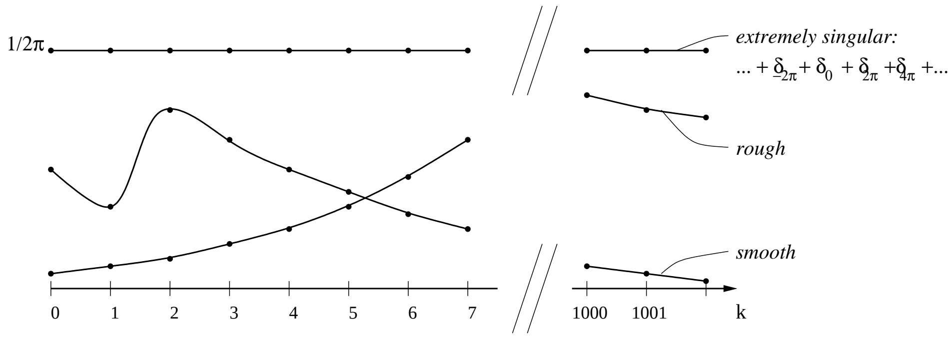
which is 2π-period.

Remark: Steep gradients $\Longleftrightarrow$ High frequencies



$$\frac{d}{dt} b_k \sin kt = b_k k \cos kt$$

$b_k k$ can be large if b_k small. This means $b_k k$ gives little contributions to $f(t)$ and big contributions to $f'(t)$



More general: $f : \mathbb{R} \longrightarrow \mathbb{C}$, 2π-periodic.

If f is "regular enough", then

1. $f(t) = \displaystyle\sum_{k=-\infty}^{\infty} c_k e^{ikt}$, with $c_k \in \mathbb{C}$ given by

2. $c_k(f) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f(t) e^{-ikt} dt$

Central question: Under which conditions on f holds (1)?

Depends on: topology ($\sum_{k=-\infty}^{\infty}$. is a limit!)

Remark: If $f : \mathbb{R} \longrightarrow \mathbb{R}$ then $a_0 = 2c_0$, $a_k = c_k + c_{-k}$, $b_k = i(c_k + c_{-k})$.

12.2 Fourier Series of Smooth Functions

$\varphi \in C^\infty(\mathbb{R})$, φ 2π-periodic

Important estimate: $\quad |c_k(\varphi)| \;\le\; \frac{1}{2\pi}\,\frac{1}{|k|^n}\;\|\varphi^{(n)}\|_\infty, \quad k \neq 0$

for $n = 1$ (in general: induction)

$$\begin{aligned} c_k(\varphi) &= \frac{1}{2\pi}\int_0^{2\pi}\varphi(t)e^{-ikt}dt \\ &= \underbrace{-\frac{1}{2\pi ik}\varphi(t)e^{-ikt}dt\Big|_0^{2\pi}}_{=0 \text{ since } \varphi(0)=\varphi(2\pi)!} + \frac{1}{2\pi ik}\int_0^{2\pi}\varphi'(t)e^{-ikt}dt \end{aligned}$$

Yields:

$t \longrightarrow \sum_{k=-\infty}^{\infty} c_k(\varphi)e^{ikt} \in C^\infty(\mathbb{R})$, absolutely convergence uniformly in t.

Question: $\varphi(t) = \sum_{k=-\infty}^{\infty} c_k(\varphi)e^{ikt}$?

Yes, if $\varphi(t) = \sum_{l=-N}^{N} \alpha_l e^{ilt}$ since $c_k(e^{ilt}) = \delta_{kl}$, so that $c_k(\varphi) = \alpha_k$

In general: Use Weierstraß approximation theorem.

For $\epsilon > 0, \exists P_N(t) = \sum_{l=-N}^{N} \alpha_l e^{ilt} : \|P_N - \varphi\|_\infty, \|P_N'' - \varphi''\|_\infty < \epsilon$

Then

$$\left|\varphi(t) - \sum_{k=-\infty}^{\infty} c_k(\varphi)e^{ikt}\right| \leq \left|\varphi(t) - P_N(t)\right| + \left|\sum_{k=-\infty}^{\infty} (c_k(\varphi) - c_k(P_N))e^{ikt}\right|$$

Note: $|c_k(\varphi) - c_k(P_N)| = |c_k(\varphi - P_N)| \leq \dfrac{1}{2\pi k^2}\|\varphi'' - P_N''\| \leq \dfrac{\epsilon}{2\pi k^2}$

Result: $\varphi \in C^\infty(\mathbb{R})$, 2π-periodic

$\Longrightarrow \varphi = \displaystyle\sum_{k=-\infty}^{\infty} c_k(\varphi)e^{ikt}$ absolutely convergence uniformly in t.

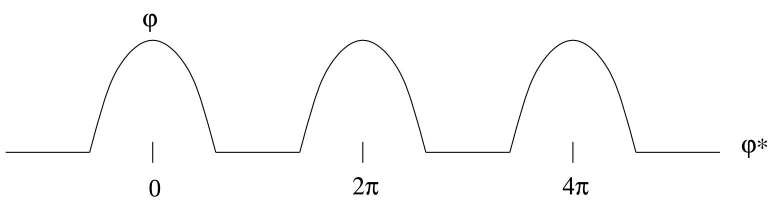
12.3 Fourier Series of Distributions

Step 1: Write $\displaystyle\int_0^{2\pi} f(t)e^{-ikt}$ as dual pairing.

$\varphi \in D(\mathbb{R})$, $f \in \mathbb{L}_{loc}'$, f 2π-periodic

$$\begin{aligned}
\langle f, \varphi\rangle &= \int_{-\infty}^{\infty} f(t)\varphi(t)dt \\
&= \sum_{k=-\infty}^{\infty} \int_0^{2\pi} \underbrace{f(t+2\pi k)}_{f(t)}\varphi(t+2\pi k)dt \\
&= \int_0^{2\pi} f(t) \underbrace{\sum_{k=-\infty}^{\infty} \varphi(t+2\pi k)dt}_{\varphi^*(t)}
\end{aligned}$$

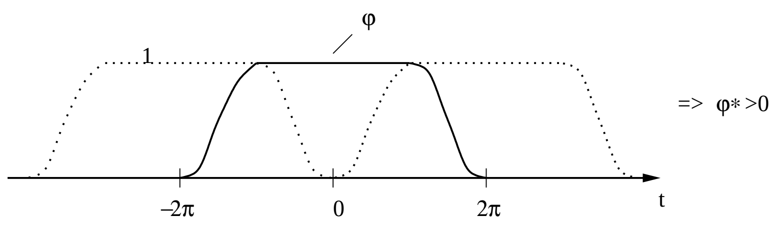
Note: sum is finite $\forall t$.



also: $\varphi^*(t+2\pi)=\varphi(t)$

observe $\varphi^*(t+2\pi)=\varphi(t) \Longrightarrow \langle f,\varphi\rangle = c_k(\varphi^*)$

Special case:



Define: $v := \dfrac{\varphi}{\varphi^*} \in D(\mathbb{R})$

Note: $v^*(t) := \sum_{k\in\mathbb{Z}} \dfrac{\varphi(t+2\pi k)}{\varphi^*(t+2\pi k)} = \dfrac{1}{\varphi^*(t)}\sum_{k\in\mathbb{Z}}\varphi(t+2\pi k) = \dfrac{\varphi^*(t)}{\varphi^*(t)} = 1$

Then: $c_k(f) = \dfrac{1}{2\pi}\langle f(t), v(t)e^{-ikt}\rangle_t$

Definition:
Let $S \in D'(\mathbb{R})$ be 2π-periodic and $v \in D(\mathbb{R})$ with $v^*=1$.
Then $c_k(S) = \dfrac{1}{2\pi}\langle S(t), v(t)e^{-ikt}\rangle_t$, $k\in\mathbb{Z}$.

Note: Definition independent of v.

Let $\bar{v}$ be another test function with $\bar{v}^*=1$

We know $S=\lim_{\epsilon\to 0} S_\epsilon$, $S_\epsilon = S * m_\epsilon \in D(\mathbb{R})$ and 2π-periodic.

$$\begin{aligned} S*m_\epsilon &= \langle S(\sigma), m_\epsilon(t+2\pi-\sigma)\rangle_\sigma \\ &= \langle S(\sigma+2\pi), m_\epsilon(t-\sigma)\rangle_\sigma \\ &= \langle S(\sigma), m_\epsilon(t-\sigma)\rangle_\sigma \\ &= S*m_\epsilon(t) \end{aligned}$$

$$\begin{aligned}
\frac{1}{2\pi}\langle S, ve^{-ikt}\rangle &= \lim_{\epsilon\to 0}\langle S_\epsilon, ve^{-ikt}\rangle \\
&= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} S_\epsilon(t)e^{-ikt}dt \\
&= \lim_{\epsilon\to 0}\langle S_\epsilon, \bar{v}e^{-ikt}\rangle \\
&= \frac{1}{2\pi}\langle S, \bar{v}e^{-ikt}\rangle
\end{aligned}$$

Step 2: Estimate $c_k(S)$

From Chapter 8: For $K \subset\subset \mathbb{R}$ there exists $m \in \mathbb{N}, C > 0$ such that

$$|\langle S, \varphi\rangle| \leq C \max_{|\alpha|\leq m} |\nabla^\alpha \varphi|$$

whenever *supp* $\varphi \subset K$.

Use $\varphi = \frac{1}{2\pi}v(t)e^{-ikt}$, and $K \supset$ *supp* v, then $|c_k(S)| = |\langle S, \varphi\rangle|$.

Note $\left|(\frac{d}{dt})^j[v(t)e^{-ikt}]\right| \leq \bar{C}|k|^j$.

$\longrightarrow \quad |c_k(S)| \leq \tilde{C}|k|^m \quad k \neq 0$

Step 3: Convergence of $\displaystyle\sum_{k=-\infty}^{\infty} c_k(S)e^{ikt}$

Assume $c_k \in \mathbb{C}$ satisfy $|c_k| \leq C|k|^m \quad k \neq 0$.

Then $\displaystyle\sum_{k\in\mathbb{Z}} c_k e^{ikt} \in D'(\mathbb{R})$

Proof:

$$\left|\frac{c_k}{(ik)^{m+2}}e^{ikt}\right| \leq \frac{C}{k^2} \quad k \neq 0$$

$$\implies \quad f(t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} \frac{c_k}{(ik)^{m+2}} e^{ikt} \quad \in C(\mathbb{R}) \text{ absolutely convergent}$$

$\implies$ convergent as series in $D'(\mathbb{R})$.

$\implies$ f regular distributions.

From 6.4, we can exchange limits with derivatives in D'!

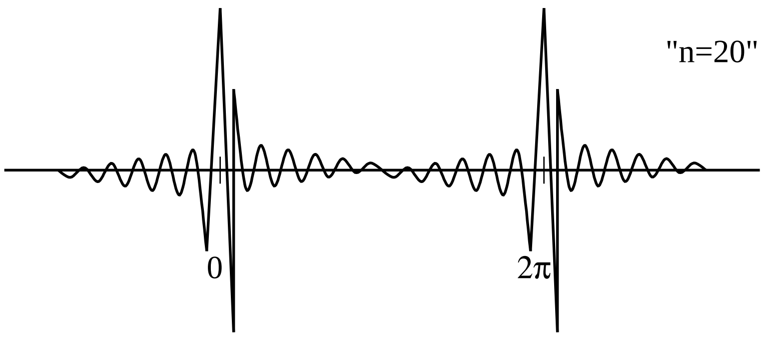
$$\implies \quad (\frac{d}{dt})^{m+2} f(t) = \sum_{k \in \mathbb{Z}\setminus\{0\}} c_k e^{ikt} \text{ convergence in } D'.$$

■

Step 4: Example: *white noise*, every frequency has same amplitude

$$c_k = \frac{1}{2\pi} \qquad \iff \qquad a_k = \frac{1}{\pi},\ b_k = 0$$

$$\sum_{k \in \mathbb{Z}} c_k e^{ikt} = \frac{1}{\pi}(\frac{1}{2} + \cos t + \cos 2t + \cos 3t + \dots)$$



concentrates at $-2\pi, 0, 2\pi, 4\pi, \dots$

$$
\begin{aligned}
\langle \sum e^{ikt}, \varphi \rangle &= \sum_{k\in\mathbb{Z}} \langle e^{ikt}, \varphi \rangle \qquad \text{convergent series in } D' \\
&= \sum_{k\in\mathbb{Z}} 2\pi c_{-k}(\varphi^*) \\
&= 2\pi \sum_{k\in\mathbb{Z}} c_k(\varphi^*) e^{ik.0} \\
&= 2\pi\varphi^*(0) \\
&= 2\pi \sum_{k\in\mathbb{Z}} \varphi(2\pi k) \\
&= \langle 2\pi\delta_{comb}, \varphi \rangle
\end{aligned}
$$

<u>Result:</u>

$$\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} = \delta_{comb}(t) = \sum_{k\in\mathbb{Z}} \delta(t+2\pi k)$$

Step 5: Main result: $S(t) = \sum_{k\in\mathbb{Z}} c_k(S)e^{ikt}$

Let $\varphi_k(t) = ve^{ikt}$ then $S * \varphi \in C^\infty(\mathbb{R})$ with $\varphi \in D(\mathbb{R})$.

$$
\begin{aligned}
S * \varphi_k(t) &= \langle S(\tau), v(t-\tau)e^{ik(t-\tau)} \rangle \\
&= e^{ikt}\langle S(\tau), \bar{v}(\tau)e^{-ik\tau} \rangle \qquad \bar{v}^* = 1 \\
&= e^{ikt} 2\pi c_k(S)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} c_k(S)e^{ikt} &= \sum_{k\in\mathbb{Z}} \frac{1}{2\pi}(S * \varphi_k(t))(t) \\
&= S * (\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \varphi_k)(t)
\end{aligned}
$$

Note: $\frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \varphi_k(t) = \frac{v(t)}{2\pi}\sum_{k\in\mathbb{Z}} e^{ikt} = v(t)\delta_{comb}(t)$

with v as in Step 1: $v(0) = 1 = \delta(t)$, $v(2\pi k) = 0$, $k \neq 0$.

$\blacksquare$

Result:

Let $S \in D'(\mathbb{R})$ 2π-period.

Then $S(t) = \sum_{k \in \mathbb{Z}} c_k(S) e^{ikt}$

with $c_k(S) = \frac{1}{2\pi} \langle S(t), v(t) e^{-ikt} \rangle,\ v \in D(\mathbb{R}),\ v^* = 1.$

Chapter 13

Fourier Transform

13.1 Basic Idea

Extend Fourier-series to non-periodic signals

Observe: In reality all signals are strictly speaking non-periodic because

they <u>*start*</u> at some point in time and <u>*end*</u> after some time interval.

Trick: Consider non-periodic signal as signal with period $2N$ and

$N \to \infty$.

Mathematically...

Let $f : R \to R$ be a smooth function with compact support.

(the signal)

- Take N so large that supp $f \subset [-N, N]$

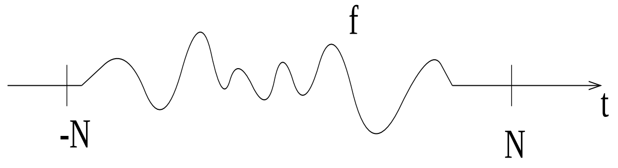


Figure 13.1:

- Go over to " periodification " f_N of f
  $f_N(t) := f(t-2Nk) \qquad \text{for } t \in [N(2k-1), N(2k+1)] \qquad k \in Z$
  Obviously f_N is $2N$ periodic.
- To use Fourier series representation, go over to $2\pi - periodic$ version.
  $\hat{f}_N(s) := f_N(\frac{2N}{2\pi}s) = f_N(\frac{N}{\pi}s)$
- Represent $\hat{f}_N$ by a Fourier series
  $\hat{f}_N(s) = \sum_{k\in Z} C_k(\hat{f}_N)e^{iks}$
  $C_k(\hat{f}_N) = \frac{1}{2\pi}\int_0^{2\pi}\hat{f}_N(s)e^{-iks}ds$
  $= \frac{1}{2\pi}\int_{-\pi}^{\pi} f_N(\frac{N}{\pi}s)e^{-iks}ds \qquad$ since $\hat{f}_N$ is $2\pi - periodic$
- Go back to old function

$t = \frac{N}{\pi}s$

$C_k(\hat{f}_N) = \frac{1}{2N}\int_{-N}^{N} f_N(t)e^{-ik\frac{\pi}{N}t}dt$

$= \frac{1}{2N}\int_R f(t)e^{-i\omega_k t}dt \qquad$ where $suppf \subset [-N, N]$ and $\omega_k = k\frac{\pi}{N}$

$$f_N(t) = \hat{f}_N(\tfrac{\pi}{N}t) = \sum_{k\in Z}\frac{e^{i\omega_k t}}{2N}\int_R f(\tau)e^{-i\omega_k\tau}d\tau$$

Note that $|\omega_k - \omega_{k+1}| := \frac{\pi}{N} =: \Delta\omega$

$$f_N(t) = \frac{1}{2\pi}\sum_{k\in Z}(e^{i\omega_k t}\int_R f(\tau)e^{-i\omega_k\tau}d\tau)\Delta\omega$$

In the limit $N\infty$ we certainly get

$$f_N(t) \to f(t) \text{ as } N \to \infty$$

and the right hand side converges to an integral

$$f(t) = \frac{1}{2\pi}\int_R e^{i\omega t}\hat{f}(\omega)d\omega$$

$$\hat{f}(\omega) := \int_R f(\tau)e^{-i\omega\tau}d\tau$$

Interpretation: The signal f can be decomposed in sin/cos where the

frequency ω appears with the amplitude $\hat{f}(\omega)$

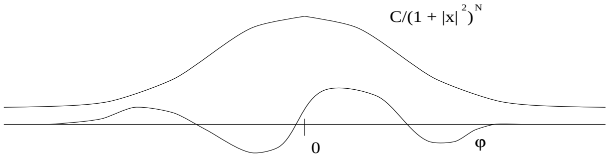


Figure 13.2:

13.2 Fourier Transform of Rapidly Decaying Test Functions

$\varphi \in C^\infty(R^d)$ is rapidly decaying if

$P_N(\varphi) := Sup_{|\alpha|\leq N} Sup_{x\in R^d}(1+|x|^2)^N |D^\alpha \varphi(x)| < \infty \qquad \forall N \in N_0$

$\mathcal{J}(R^d)$: rapidly decaying test functions.

$\varphi \in \mathcal{J}$ decays faster than polynomials :

$$|D^\alpha \varphi(x)| \leq \frac{P_N(\varphi)}{(1+|x|^2)^N} \qquad \forall |\alpha| \leq N$$

Note: $\mathcal{J}(R^d)$ is a vector space

$\mathcal{D}(R^d) \subset \mathcal{J}(R^d)$

- Convergence in $\mathcal{J}(R^d)$

$(\varphi_k)_{k\in N} \subset \mathcal{J}(R^d)$

$\lim\limits_{k\to\infty} \varphi_k = 0 \qquad$ if $\qquad P_N(\varphi_k) \to 0$ as $k \to \infty \qquad \forall N \in N$

- Fourier Transform in $\mathcal{J}(R^d)$

$\mathcal{F}(\varphi)(\xi) := \hat{\varphi}(\xi) := \int_{R^d} e^{-i\xi x}\varphi(x)dx$

Main result :

$\mathcal{F} : \mathcal{J}(R^d) \to \mathcal{J}(R^d)$

is a linear homeomorphism (bijective, $\mathcal{F}, \mathcal{F}^{-l}$ continuous)

The inverse is given by

$$\mathcal{F}^{-l}(\psi)(x) = \frac{1}{(2\pi)^d}\int_{R^d} e^{i\xi x}\psi(\xi)d\xi$$

Main Properties :

1) Derivatives $\leftrightarrow$ Multiplied by polynomials

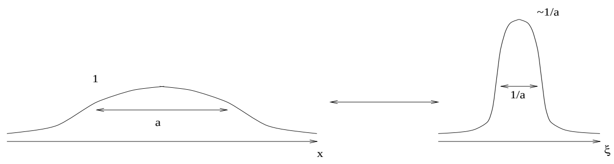


Figure 13.3:

$$\mathcal{F}(\nabla^\alpha \varphi)(\xi) = (i\xi)^\alpha \mathcal{F}(\varphi)(\xi)$$

$$\mathcal{F}(x^\alpha \varphi)(\xi) = (i\nabla)^\alpha \mathcal{F}(\varphi)(\xi)$$

2) Convolution $\leftrightarrow$ Product

$$\mathcal{F}(\varphi * \psi) = \mathcal{F}(\varphi)\mathcal{F}(\psi)$$

$$\mathcal{F}(\varphi.\psi) = (2\pi)^{-d}\mathcal{F}(\varphi) * \mathcal{F}(\psi)$$

3) Duality relation

$$\int \varphi \mathcal{F}(\psi) dx = \int \psi \mathcal{F}(\varphi) dx$$

4) $\mathcal{L}^2$ -isometry

$$< \varphi, \psi >_{\mathcal{L}^2} = \int \varphi \bar{\psi} dx =< \mathcal{F}(\varphi), \mathcal{F}(\psi) >_{\mathcal{L}^2}$$

5) Gaussian $\leftrightarrow$ $Gaussian$

$$\mathcal{F}(e^{-a|x|^2}) = (\tfrac{\pi}{a})^{\frac{d}{2}} e^{-\frac{|\xi|^2}{4a}}$$

Remarks:

a) $\quad d = 1 \qquad \mathcal{F}(\varphi')(\xi) = \int_R \varphi'(x) e^{-ix\xi} dx = (-i\xi) e^{-ix\xi} \varphi(x)|_{x=-\infty}^{x=+\infty}$

$$(-i\xi) \int \varphi(x) e^{-ix\xi} dx = (i\xi)\mathcal{F}(\varphi)$$

We can introduce "half derivatives" (pseudo differential operators)

$$(\tfrac{d}{dx})^{\frac{1}{2}} \varphi := \mathcal{F}^{-l}(\sqrt{i\xi}\hat{\varphi})$$

Note : $(\frac{d}{dx})^{\frac{1}{2}} (\frac{d}{dx})^{\frac{1}{2}} = \frac{d}{dx}$

b) $\quad$ Smoothness $\leftrightarrow$ Decay

$$|\xi|^n |\mathcal{F}(\varphi)(\xi)| \leq \int |\varphi^{(n)}(x)| dx$$

$$\rightsquigarrow \qquad |(\mathcal{F}\varphi)(\xi)| \leq \frac{||\varphi^{(n)}||_{\mathcal{L}^1}}{|\xi|^n}$$

fast decay if φ is very smooth.

13.3 Tempered Distributions

$\mathcal{J}'(R^d)$: Continuous linear functionals on $\mathcal{J}(R^d)$

= tempered distribution

Example : $f(x) = x^\alpha$ (polynomial growth)

$< T_f, \varphi >:= \int f(x)\varphi(x)dx$ where $\varphi(x)$ drops faster than any polynomial $\rightarrow$ well-defined.....

More general : $f \in \mathcal{L}^1_{loc}(R^d)$

$$|f(x)| \leq P(x)(1+|x|)^m \qquad P \in \mathcal{L}^1, P \geq 0$$

Then $T_f \in \mathcal{J}'(R^d)$

Also : $\delta \in \mathcal{J}'(R^d)$ since $< \delta, \varphi >= \varphi(0)$ is continuous on $\mathcal{J}$

More general : $\varepsilon'(R^d) \hookrightarrow \mathcal{J}'(R^d) \hookrightarrow \mathcal{D}'(R^d)$

Because : $\mathcal{D}(R^d) \hookrightarrow \mathcal{J}(R^d) \hookrightarrow \varepsilon(R^d)$

13.3.1 Fourier Transformation of Tempered Distributions

Use (3) $< \varphi, \mathcal{F}(\psi) >=< \psi, \mathcal{F}(\varphi) >$ $\qquad \varphi, \psi \in \mathcal{J}(R^d)$

To motivate

$\underline{Definition}$: Let $S \in \mathcal{J}'(R^d)$. Then $\mathcal{F}(S)$ is defined by

$$< \mathcal{F}(S), \varphi >:=< S, \mathcal{F}(\varphi) > \qquad \forall \varphi \in \mathcal{J}(R^d)$$

Main result: $\mathcal{F} : \mathcal{J}'(R^d) \rightarrow \mathcal{J}'(R^d)$

is a linear homeomorphism.

(w.r.t convergence in $\mathcal{J}'$: $S_n \rightarrow 0 \iff^{Def} < S_n, \varphi > \rightarrow 0 \qquad \forall \varphi \in \mathcal{J}$)

The inverse is given by $< \mathcal{F}^{-l}(S), \varphi >=< S, \mathcal{F}^{-l}(\varphi) >$

1) holds in $\mathcal{J}'$

2) holds for convolution with test functions

3) holds by definition

f =

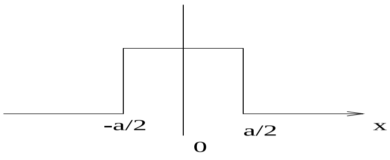


Figure 13.4:

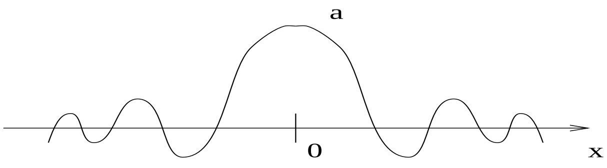


Figure 13.5:

Examples :

1) $< \mathcal{F}(\delta), \varphi > = < \delta, \mathcal{F}(\varphi) > = \mathcal{F}(\varphi)(0) = \int \varphi(x)dx$

$= < 1, \varphi >$

$\Rightarrow \underline{\mathcal{F}\delta = 1}$ (white noise = every frequency, same amplitude)

$< \mathcal{F}(1), \varphi > = < 1, \mathcal{F}(\varphi) > = \int \mathcal{F}(\varphi)(\xi) e^{i\xi \cdot 0} d\xi = (2\pi)^d \varphi(0)$

$= (2\pi)^d < \delta, \varphi >$

$\underline{\mathcal{F}1 = (2\pi)^d \delta}$

2)

$< \mathcal{F}(f), \varphi > = < f, \mathcal{F}(\varphi) > = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_R \varphi(x) e^{-ix\xi}$ dx $d\xi$

$= \int_R \varphi(x) [\frac{1}{-ix} e^{-ix\xi}]_{\xi=-\frac{a}{2}}^{\xi=\frac{a}{2}} dx$

Note :

$[\frac{1}{-ix} e^{-ix\xi}]_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{2}{x} sin(\frac{a}{2}x) =$ a $sinc(\frac{a}{2}x)$

where $sinc(x) = \frac{sinx}{x}$

$\rightsquigarrow$ $< \mathcal{F}(f), \varphi > = <$a $sinc(\frac{a}{2}x), \varphi(x) >_x$

$\rightsquigarrow$ $\mathcal{F}(f)(x) =$ a $sinc(\frac{a}{2}x)$

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