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Optimierung
4. Übungsblatt

Unconstrained optimization.

Until now, we looked for local minimal points x^* of a sufficiently smooth, real-valued function f in $\Omega = \mathbb{R}^n$:

$$x^* = \arg\min_{x\in\Omega} f(x).$$

By differential calculus, we immediately received as a necessary "first-order" condition:

$$f(x^*) \leq f(x) \text{ for all } x \in B_\epsilon(x^*) \qquad \Longrightarrow \qquad \forall x \in \Omega : \langle \nabla f(x^*), x\rangle = 0.$$

Optimization with boundary constraints.

If Ω is closed, the situation is slightly more complicated: Local minimizers on the boundary are possible, but here the gradient condition is not a necessary criterion.

Let $\Omega \subseteq \mathbb{R}^n$ a closed interval, i.e. there are $a_i, b_i \in \mathbb{R}$ $(i = 1, ..., n)$ such that

$$\Omega = \prod_{i=1}^{n} [a_i, b_i] = \{x \in \mathbb{R}^n \mid \forall i = 1, ..., n : a_i \leq x \leq b_i\}.$$

□ Exercise 12

Let $f \in \mathcal{C}^2(\Omega, \mathbb{R})$. Notice that $\nabla f : \Omega^\circ \to \mathbb{R}^n$ can be extended to the boundary of Ω since $f \in \mathcal{C}^2$ implies that ∇f is Lipschitz continuous on Ω°.

Further, let $x^* \in \Omega$ a local minimizer of f, i.e.

$$\exists \epsilon > 0 : \forall x \in B_\epsilon(x^*) \cap \Omega : f(x^*) \leq f(x).$$

Prove that the following modified first-order condition holds:

$$\forall x \in \Omega : \langle \nabla f(x^*), x - x^*\rangle \geq 0.$$

Any x^* that fulfills this condition is called *stationary point* of f.

Canonical projection on the domain.

The *canonical projection* of $\mathbb{R}^n$ on Ω is defined by

$$\Big(\mathbb{P}(x)\Big)_i := \begin{cases} a_i & \text{if } x_i \leq a_i \\ x_i & \text{if } x_i \in [a_i, b_i] \\ b_i & \text{if } x_i \geq b_i \end{cases}$$

☐ **Exercise 13**

Let L the Lipschitz constant for ∇f. Define

$$x(\lambda) := \mathbb{P}(x - \lambda \nabla f(x)).$$

Prove that the following ***modified Armijo condition*** holds for all $\lambda \in \left(0, \frac{2(1-\alpha)}{L}\right]$:

$$f(x(\lambda)) - f(x) \leq -\frac{\alpha}{\lambda} ||x - x(\lambda)||^2.$$

Hint: The following ansatz with the fundamental theorem of calculus may be helpful:

$$f(x(\lambda)) - f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\Big(x - t\big(x - x(\lambda)\big)\Big) \mathrm{d}t.$$

Hint: You may make use of the following formula (without proof):

$$\langle x - x(\lambda), x(\lambda) - x + \lambda \nabla f(x) \rangle \geq 0.$$

The Projected Gradient Method.

We modify the Steepest Descent Algorithm with the modified Armijo stepsize strategy such that the algorithm can be applied for the situation above:

Algorithm. (Projected Gradient Method)
while some termination condition is not fulfilled
  while modified Armijo condition is not fulfilled
    set $\lambda = \frac{\lambda}{2}$
  end
  set $x = x(\lambda)$
end

Our objective is to prove that the generated iteration sequence has a convergent subsequence which converges towards a stationary point of f, cp. Satz 3.8 in the lecture notes.

☐ **Exercise 14**

Let $(x_n)_{n\in\mathbb{N}}$ an iteration sequence created by the Projected Gradient Algorithm.

1. Show that $(f(x_n))_{n\in\mathbb{N}}$ converges.
2. Show that $(x_n)_{n\in\mathbb{N}}$ has at least one convergent subsequence.
3. Furthermore, show that all accumulation points of $(x_n)_{n\in\mathbb{N}}$ are stationary points of f.
4. Show that x^* is a stationary point of f if and only if $x^* = \mathbb{P}(x^* - \lambda \nabla f(x^*))$ holds.