INF2340: A CRASH COURSE IN NUMERICAL METHODS FOR CONSERVATION LAWS

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This note is devoted to the numerical solution of hyperbolic conservation laws, giving a crash course in the most important mathematical and numerical concepts used to build efficient computational methods. The note is intended to be a complement to the material covered in the INF2340–lectures and in the lecture notes by R.J. LeVeque.

1. Hyperbolic conservation laws

The term "hyperbolic conservation laws" usually denotes a first-order, quasilinear partial differential equation on the following form (in one spatial dimension)

$$u_t + f(u)_x = 0. \tag{1}$$

Here u is some conserved quantity (scalar or vector) and $f(u)$ is a flux function. A conservation law usually arises from a more fundamental physical law on integral form. In one spatial dimension, this law typically reads

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f\big(u(x_1,t)\big) - f\big(u(x_2,t)\big). \tag{2}$$

The physical law states that the rate of change of quantity u within $[x_1, x_2]$ equals the flux across the boundaries $x = x_1$ and $x = x_2$. The partial differential equation (1) then follows under additional regularity assumptions on u.

The problem typically encountered in conservation laws is the initial-value problem for (1),

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{3}$$

which is often referred to as the Cauchy problem, For a nonlinear flux function f, this equation may develop discontinuities—singularities in the first-order derivatives—in finite time even for smooth initial data. This means that the solution of (3) is understood in the weak form,

$$\int_0^\infty \int_{\mathbb{R}} \big(u\phi_t + f(u)\phi_x\big)\,dtdx = \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx. \tag{4}$$

Here $\phi(x,t)$ is a smooth test function possessing all the necessary derivatives. The function $\phi(x,t)$ is also assumed to have *compact support*, meaning that it vanishes outside a bounded region in the (x,t)-plane.

Solutions defined by the weak form (4) are not necessarily unique. The solution concept must therefore be extended to include additional admissibility conditions to single out the correct solution among several possible candidates satisfying the weak form. A classical method to obtain uniqueness is to add a regularising second-order term to (1), giving an equation

$$u^\varepsilon_t + f(u^\varepsilon)_x = \varepsilon u^\varepsilon_{xx},$$

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that has smooth solutions. Then the unique solution of (1) is defined as the limit of $u^\varepsilon(x,t)$ as ε tends to zero. Physically, this corresponds to adding viscosity, and the method is therefore called the *vanishing viscosity method*. Since the solutions $u^\varepsilon(x,t)$ are smooth, classical analysis coupled with careful limit arguments can be used to show existence, uniqueness and stability of the solution of (1). This was first done by Kruzkov [8], whose seminal work on 'doubling of the variables' has had a tremendous impact on the development of modern theory for nonlinear partial differential equations.

Working with limits of viscous solutions is not what we want. Instead we impose other conditions. There are several ways to impose such admissibility conditions. In this book we will use the concept of *entropy functions*, which is motivated from thermodynamic conditions in gas dynamics. To this end, we introduce a (convex) entropy function $\eta(u)$ and a corresponding entropy flux $\psi(u)$ and require that an admissible weak solution u must satisfy the *entropy condition*

$$\eta(u)_t + \psi(u)_x \leq 0, \tag{5}$$

which must be interpreted in the weak sense as

$$\int_0^\infty \int_{\mathbb{R}} \big(\eta(u)\phi_t + \psi(u)\phi_x\big)\, dtdx + \int_{\mathbb{R}} \eta(u_0(x))\phi(x,0)\, dx \geq 0. \tag{6}$$

The functions η and ψ are called an entropy pair and satisfy the compatibility condition

$$\psi'(u) = \eta'(u)f'(u).$$

The existence of such entropy pairs is not obvious for a general system of conservation laws, but such pairs exist and have a clear physical interpretation for several important systems of equations, for instance in gas dynamics. For scalar equations it can be shown that all entropy pairs with convex η are equivalent. A common choice is the so-called Kruzkov entropy pair,

$$\int_0^\infty \int_{\mathbb{R}} \big(|u-k|\phi_t + \operatorname{sign}(u-k)[f(u)-f(k)]\phi_x\big)\, dtdx + \int_{\mathbb{R}} |u_0(x)-k|\phi(x,0)\, dx \geq 0. \tag{7}$$

We say that $u(x,t)$ is an *entropy weak solution* of (1) if it satisfies (7) for all real numbers k and suitable test functions $\phi(x,t) > 0$.

2. Finite-volume methods

Finite-difference methods use discrete differences to approximate the derivatives in a partial differential equation. This gives discrete evolution equations for a set of point values approximating the true solution of the PDE. Once a discontinuity arises in the hyperbolic conservation law, the differential equation will cease to be pointwise valid in the classical sense. Hence, it is also to be expected that classical finite-difference approximations will break down at discontinuities, causing severe problems for standard finite-difference methods. To overcome this computational problem, it turns out that instead of seeking *pointwise* solutions to (1), one should look for solutions of the more fundamental integral form (2). To this end, we break the domain $[x_1, x_2]$ into a set of subdomains — which we call finite volumes or grid cells — and seek approximations to the *global* solution u in terms of a discrete set of *cell average* defined over each grid cell; that is, we seek approximations to $\int u(x,t)\, dx$ over each grid cell.

There is a close relation between finite-difference and finite-volume methods since the formula of a specific finite-volume method in some cases may be interpreted directly as a finite-difference approximation to the underlying differential equation. However, the underlying

principles are fundamentally different. Finite difference methods evolve a discrete set of point values by approximating (1). Finite-volume methods evolve *globally defined solutions* as given by (2) and *realize* them in terms of a discrete set of cell averages. The evolution of globally defined solutions is the key to the success of modern methods for hyperbolic conservation laws. There are many good books describing such methods. We can recommend the books by Godlewski and Raviart [4], Kröner [7], LeVeque [13, 14], and Toro [23].

3. Conservative methods

The starting point for a finite-volume method for (1) is the cell-average defined by

$$u_i^n = \frac{1}{\Delta x\, i} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t_n)\, dx.$$

These cell averages are usually evolved in time by an explicit time-marching method, obtained by integrating (2) in time

$$u_i^{n+1} - u_i^n = \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(u(x_{i-1/2}, t))\, dt - \frac{1}{\Delta x} \int_{t_n}^{t_{n+1}} f(u(x_{i+1/2}, t))\, dt. \tag{8}$$

Generally, we will not be able to compute the flux integrals exactly, since $u(x_{i\pm1/2}, t)$ varies with time and is in general unknown. However, the equation suggests that the numerical method should be of the form

$$u_i^{n+1} = u_i^n - \lambda\big(F_{i+1/2}^n - F_{i-1/2}^n\big), \tag{9}$$

where $\lambda = \Delta t/\Delta x$ and $F_{i\pm1/2}^n$ is some approximation to the average flux over each cell interface,

$$F_{i\pm1/2}^n \approx \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u(x_{i-1/2}, t))\, dt.$$

Any numerical method on this form will generally be *conservative*. To see this, we can sum the equation over all i. The flux terms will cancel in pairs, and we are left with

$$\sum_{i=-M}^{M} u_i^{n+1} = \sum_{i=-M}^{M} u_i^n - \lambda\big(F_{M+1/2}^n - F_{-M-1/2}^n\big).$$

The two flux terms vanish if we assume either periodic boundary conditions or that $u(x, t)$ approaches the same constant value as $x \to \pm\infty$. Thus, the numerical method conserves the quantity u, i.e.,

$$\int u^n(x)\, dx = \int u_0(x)\, dx.$$

Hyperbolic conservation laws have finite speed of propagation, unless they degenerate in some form. It is therefore natural to assume that the average fluxes are given in terms of their neighboring cell averages; that is,

$$F_{i+1/2}^n = F\big(u_{i-p}^n, \ldots, u_{i+q}^n\big).$$

The function F is called the *numerical flux function* and will be referred to by the abbreviation $F(u^n; i)$.

4. A few classical schemes

We have now gone through the basic underlying principles for the design of schemes for conservation laws. It is therefore time to show some examples of such schemes.

The simplest example is the upwind scheme. If $f'(u) \geq 0$, the scheme has numerical flux $F(u^n; i) = f(u_i^n)$ and reads (with $\lambda = \Delta t/\Delta x$)

$$u_i^{n+1} = u_i^n - \lambda\big[f(u_i^n) - f(u_{i-1}^n)\big] \tag{10}$$

Similarly, if $f'(u) \leq 0$, the upwind scheme takes the form

$$u_i^{n+1} = u_i^n - \lambda\big[f(u_{i+1}^n) - f(u_i^n)\big]$$

In either case, the upwind scheme is a two-point scheme based upon one-sided differences in the so-called *upwind* direction, i.e., in the direction where the information flows from. The idea of upwind-differencing is the underlying design principle behind a large number of schemes of the Godunov-upwind type, which we will return to below.

Another classical scheme is the three-point Lax–Friedrichs scheme,

$$u_i^{n+1} = \tfrac{1}{2}\big(u_{i-1}^n + u_{i+1}^n\big) - \tfrac{1}{2}\lambda\big[f(u_{i+1}^n) - f(u_{i-1}^n)\big]. \tag{11}$$

The Lax–Friedrichs scheme is based upon central differencing and is a very stable, all-purpose scheme that will always converge, although sometimes painstakingly slowly. The scheme can be written in conservation form by introducing the numerical flux

$$F(u^n; i) = \frac{1}{2\lambda}\big(u_i^n - u_{i+1}^n\big) + \frac{1}{2}\big[f(u_i^n) + f(u_{i+1}^n)\big].$$

The upwind and the Lax–Friedrichs schemes are both examples of schemes that are formally first-order in the sense that their truncation error is of order two, see Section 5. Hence the schemes will converge with order one for *smooth* solutions.

Better accuracy can be obtained if we make a better approximation to the integral in the definition of the average flux. Instead of evaluating the integral at the endpoint t_n, we can evaluate it at the midpoint $t_{n+1/2} = t_n + \frac{1}{2}\Delta t$. This gives a classical second-order method called the Richtmeyer two-step Lax–Wendroff method

$$\begin{aligned} u_{i+1/2}^{n+1/2} &= \tfrac{1}{2}\big(u_i^n + u_{i+1}^n\big) - \tfrac{1}{2}\lambda\big[f(u_{i+1}^n) - f(u_i^n)\big], \\ u_i^{n+1} &= u_i - \lambda\big[f(u_{i+1/2}^{n+1/2}) - f(u_{i-1/2}^{n+1/2})\big]. \end{aligned} \tag{12}$$

The corresponding numerical flux reads

$$F(u^n; i) = f\big(\tfrac{1}{2}(u_i^n + u_{i+1}^n) - \tfrac{1}{2}\lambda\big[f(u_{i+1}^n) - f(u_i^n)\big]\big).$$

Another popular variant is MacCormack's method

$$\begin{aligned} u_i^* &= u_i^n - \lambda\big[f(u_{i+1}^n) - f(u_i^n)\big] \\ u_i^{**} &= u_i^* - \lambda\big[f(u_i^*) - f(u_{i-1}^*)\big] \\ u_i^{n+1} &= \tfrac{1}{2}\big(u_i^n + u_i^{**}\big) \end{aligned} \tag{13}$$

which has the following numerical flux

$$F(u^n; i) = \tfrac{1}{2}f(u_{i+1}^n) + \tfrac{1}{2}f\big(u_i^n - \lambda\big[f(u_{i+1}^n) - f(u_i^n)\big]\big).$$

All the above schemes are stable under the CFL restriction

$$\lambda \max_u |f'(u)| \leq 1.$$

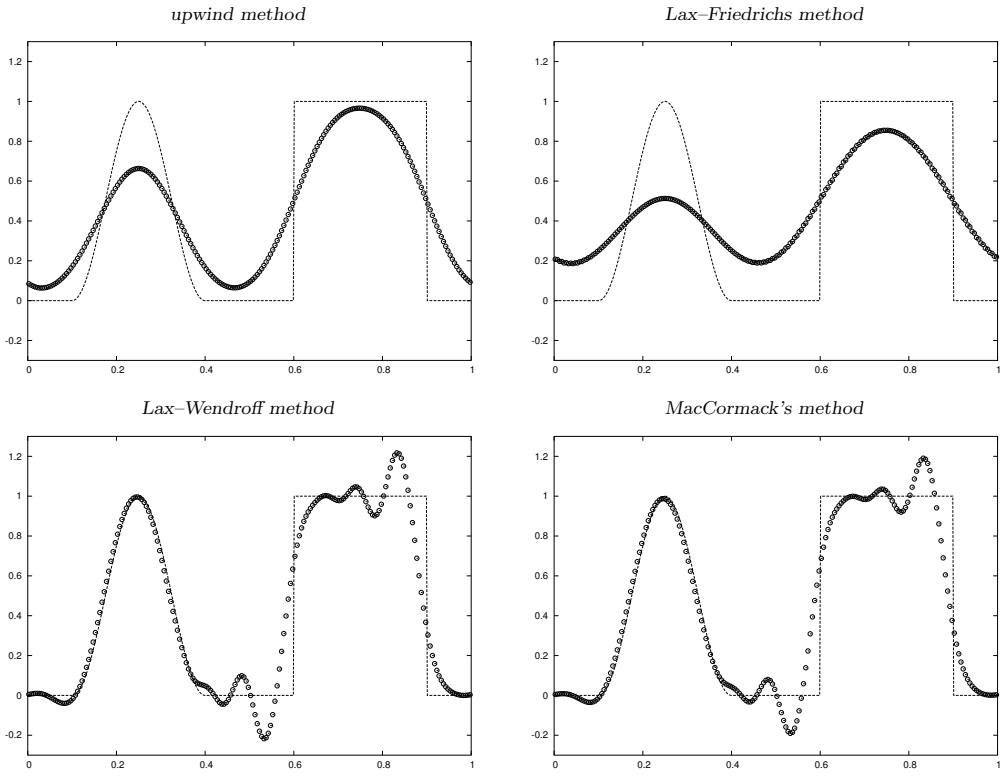

FIGURE 1. Approximate solutions at time $t = 10.0$ of the linear advection equation (14).

Let us now apply the four schemes to two examples to gain some insight in their behaviour.

Example 1. *Let us first consider a linear advection equation with periodic boundary data*

$$u_t + u_x = 0, \qquad u(x,0) = u_0(x), \qquad u(0,t) = u(1,t). \tag{14}$$

As initial data $u_0(x)$ we choose a combination of a smooth squared cosine wave and a double step function.

Figure 1 shows approximate solutions at time $t = 10.0$ computed by the four schemes on a grid with 200 nodes using a time-step restriction $\Delta t = 0.9\Delta x$. We see that the two first-order schemes smear both the smooth part and the discontinuous path of the advected profile. The second-order schemes, on the other hand, preserve the smooth profile quite accurately, but introduce spurious oscillations around the discontinuities.

Example 2. *In this example we apply the four schemes introduced above to Burgers' equation with discontinuous initial data*

$$u_t + \left(\tfrac{1}{2}u^2\right)_x = 0, \qquad u(x,0) = \begin{cases} 1, & x \le 0.1 \\ 0, & x > 0.1 \end{cases} \tag{15}$$

Burgers' equation is the archetypical example of a nonlinear equation, possessing a convex flux that may cause discontinuous shock waves to form even for smooth initial data.

Figure 2 shows approximate solutions at time $t = 0.5$ computed by all four schemes on a grid with 50 uniform grid cells and a time-step restriction $\lambda = 0.6$. Comparing the two

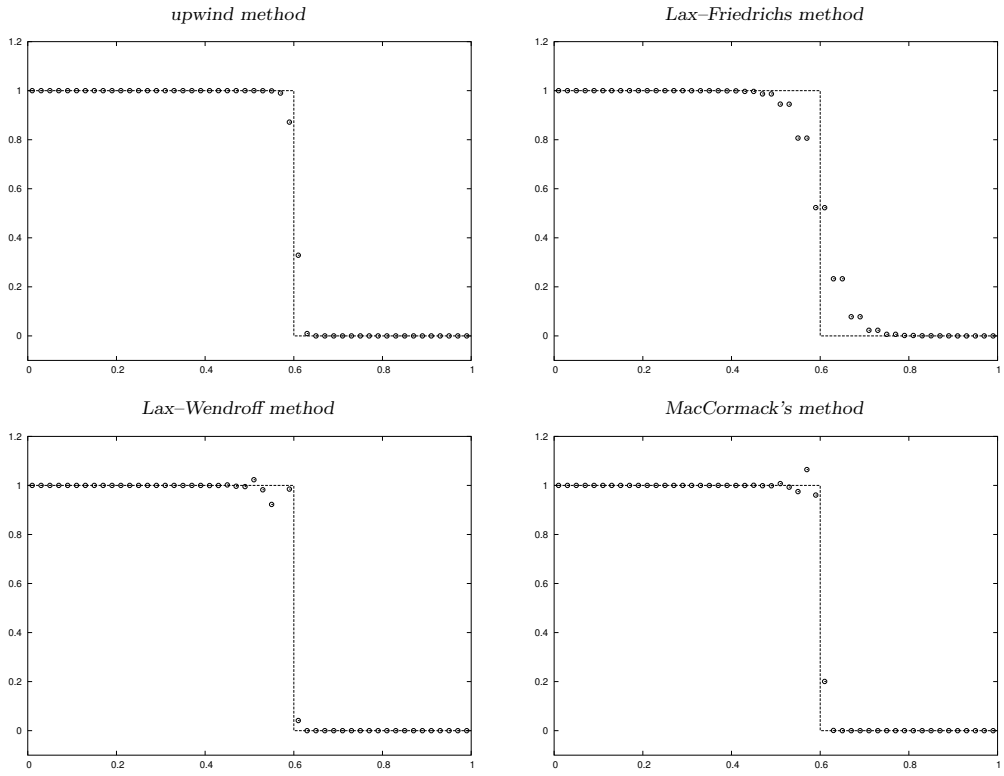

FIGURE 2. Approximate solutions at time $t = 0.5$ of the Burgers' equation (15).

first-order schemes, we see that the upwind scheme resolves the discontinuity quite sharply, whereas the Lax–Friedrichs smears it out over several grid cells. Both second-order schemes resolve the discontinuity sharply, but produce spurious oscillations downstream.

Although the two examples above were fairly simple, neither of the schemes were able to compute approximate solutions with a satisfactory resolution (except for the upwind scheme in Example 2). The first-order methods lack the resolution to prevent smooth linear waves from decaying and discontinuities to be smeared, whereas second-order methods introduce nonphysical oscillations near discontinuities. Conceptually, one could imagine a possible marriage of the two types of methods in which we try to retain the best features of each method. The resulting scheme would then have second-order (or higher) accuracy in smooth regions of the solution and at the same time have the stability of a first-order scheme where the solution is not smooth. This is a key concept underlying so-called *high-resolution* schemes. Assume now that θ_i^n is a quantity measuring the smoothness of the solution at grid cell i at time t_n such that θ_i^n is close to unity if the solution is smooth and θ_i^n is close to zero if the solution is discontinuous. Then a hybrid method with numerical flux

$$F(u^n; i) = \big(1 - \theta_i^n\big) F_L(u^n; i) + \theta_i^n F_H(u^n; i)$$

would give the desired properties. Here $F_L(u^n; i)$ denotes a low-order flux like the upwind or the Lax–Friedrichs flux and $F_H(u^n; i)$ is a high-order flux like the Lax–Wendroff or the MacCormack flux. The quantity $\theta_i^n = \theta(u^n; i)$ called a *limiter* and the method is called a *flux-limiter* method. A large number of successful high-resolution methods have been developed

based on the flux-limiting approach. A review of such methods if outside the scope of the note. We will instead retrace our steps in Section 6 and review a more geometrical framework for developing high-resolution methods. But before we do that, let us take a careful look at the schemes introduced so far.

5. Convergence of conservative methods

So far, we have established an appropriate framework for designing numerical schemes for the conservation law (1) and shown a few examples of such schemes. But how can we ensure that these schemes will compute correct approximations to the equation? Moreover, how can we be certain that a scheme will *converge* to the true solution as the discretisation parameters tend to zero?

To discuss this, we must first define what we mean by convergence. The classical way of analysing convergence is to consider the truncation error $L_{\Delta t}$ and show that this error tends to zero with the discretisation parameters; that is, $L_{\Delta t} = \mathcal{O}(\Delta t^{r+1})$ for $r > 0$, where r is said to be the order of the scheme. The truncation error for a scheme on the form (19) is defined as

$$L_{\Delta t} = u(x, t + \Delta t) - \Big(u(x,t) - \lambda\big[F(u(x,t); i) - F(u(x,t); i-1)\big]\Big).$$

We have seen above that the differential equation (1) is not valid in a pointwise sense for discontinuous solutions. Thus, the pointwise truncation error cannot be used to establish convergence. For conservation laws the truncation error only defines the *formal order* of a scheme, i.e., the order the scheme would converge with for smooth solutions.

Since solutions of conservation laws generally are taken in the weak sense (4), they are not generally unique. Pointwise errors of the form $u_{\Delta t}(x,t) - u(x,t)$ are therefore not well-defined, where $u_{\Delta t}$ is defined by an appropriate interpolation of u_i^n. Instead, we must measure the deviation in some appropriate norm. It turns out that for scalar equation, the L^1 norm is the correct norm, and we say that an approximation $u_{\Delta t}$ converges to a function u if

$$\int_0^t \|u_{\Delta t}(\cdot, t) - u(\cdot, t)\|_1 \, dt \to 0, \text{ as } \Delta t \to 0.$$

A famous theorem due to Lax and Wendroff [18] states that if $u_{\Delta t}$ is computed by a *conservative* and *consistent* scheme and if $u_{\Delta t}$ converges to a function u almost everywhere in a uniformly bounded manner, then u is a weak solution to the conservation law (1).

We verified above that any scheme on the form (19) is conservative. The scheme (9) is said to be *consistent* if

$$F(v, \ldots, v) = f(v).$$

Moreover, one generally requires the numerical flux to be Lipschitz continuous, i.e., that there is a constant L such that

$$|F(u_{i-p}, \ldots, u_{i+q}) - f(u)| \leq L \max(|u_{i-p} - u|, \ldots, |u_{i+q} - u|).$$

For linear equations the *Lax equivalence theorem* states that a scheme is convergent if it is stable and consistent. A scheme is *stable* if the errors introduced at a time step do not grow (too fast) in time. To ensure stability one generally has to impose a restriction on the time-step through a *CFL condition*, named after Courant, Friedrichs, and Lewy, who wrote one of the first papers on finite difference methods in 1928 [3]. The CFL condition states that the true domain of dependence for the PDE (1) should be contained in the domain of dependence for (9).

The Lax equivalence theorem does not hold for nonlinear equations. However, similar results hold if we can prove that the numerical scheme is contractive in some appropriate norms since this guarantees that the sequence of approximations is compact and thus convergent. A sequence $\{s_1, s_2, \dots\}$ of elements in a space $\mathcal{S}$ is compact if it contains a subsequence that converges to an element in $\mathcal{S}$.

The set of functions with bounded (total) variation is compact in L^1. The total variation of a continuous function is defined as

$$TV(v) = \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{-\infty}^{\infty} |v(x+\varepsilon) - v(x)|\, dx.$$

For a piecewise constant function the definition of the total variation simplifies to

$$TV(v) = \sum_i |v_i - v_{i-1}|.$$

This means that we can show that a conservative and consistent scheme will converge if we can verify that the corresponding sequence $\{u_{\Delta t}\}$ has uniformly bounded total variation. There are several ways to verify uniformly bounded total variation. The total variation of the exact solution of a scalar conservation law is nonincreasing with time

$$TV(u(\cdot, t)) \le TV(u(\cdot, s)), \qquad t \ge s.$$

An obvious way to ensure uniformly bounded variation is therefore to require that the scheme has the same property; that is,

$$TV(u^{n+1}) \le TV(u^n). \tag{16}$$

Any scheme that satisfies (16) is called a *total variation diminishing* method, commonly abbreviated as a TVD-method. This requirement has been a popular design principle for a large number of successful schemes, see Section 6.

In addition there are other properties that might be attractive for the scheme to fulfill

- A scheme is *monotonicity preserving* when it ensures that if the initial data u_i^0 is monotone then so is u_i^n for any n.
- A scheme is L^1 *contractive* if $\|u_{\Delta t}(\cdot, t)\|_1 \le \|u_{\Delta t}(\cdot, 0)\|_1$. The entropy weak solution of a scalar conservation law is L^1 contractive.
- A method is *monotone* if

$$u_i^n \ge v_i^n \quad \forall i \qquad \Longrightarrow \qquad u_i^{n+1} \ge v_i^{n+1} \quad \forall i.$$

For conservative and consistent methods, these properties are related as follows:

- Any monotone method is L^1 contractive.
- Any L^1 contractive method is TVD.
- Any TVD method is monotonicity preserving.

Verifying that a method is monotone is quite easy. Unfortunately, it has been proved that a monotone method is at most first order accurate.

The Lax–Wendroff and compactness theory review above can be used to verify that a scheme converges to a weak solution of the conservation law. On the other hand, the theory does not say anything of whether the limit is the correct entropy weak solution or not. However, one can show that if the scheme (9) satisfies a so-called *cell entropy condition* on the form

$$\eta(u_i^{n+1}) \le \eta(u_i^n) - \lambda\big(\Psi_{i+1/2}^n - \Psi_{i-1/2}^n\big), \tag{17}$$

then the limiting weak solution is in fact the entropy weak solution. Here Ψ is a numerical entropy flux that must be consistent with the entropy flux ψ in the same way as we required the numerical flux F to be consistent with the flux f.

6. High-resolution Godunov methods

A large number of successful high-resolution methods can be classified as Godunov methods. We have seen several examples of such methods above. In the following we will therefore introduce the general setup of Godunov-type methods in some detail, thereby retracing some of the steps used to derive the schemes in Section 3. For simplicity, the presentation is in one spatial dimension, but the same ideas applies also in several space dimensions.

We start by defining the sliding average $\bar{u}(x,t)$ of $u(\cdot,t)$, namely

$$\bar{u}(x,t) = \frac{1}{\Delta x}\int_{I(x)} u(\xi,t)\,d\xi, \qquad I(x) = \{\xi : |\xi - x| \le \tfrac{1}{2}\Delta x\}. \tag{18}$$

If we now integrate (1) over the domain $I(x)\times[t,t+\Delta t]$, we obtain an evolution equation for the sliding average $\bar{u}(x,t)$

$$\bar{u}(x,t+\Delta t) = \bar{u}(x,t) - \frac{1}{\Delta x}\int_t^{t+\Delta t}\Big[f\big(u(x+\tfrac{1}{2}\Delta x,s)\big) - f\big(u(x-\tfrac{1}{2}\Delta x,s)\big)\Big]\,ds \tag{19}$$

This equation is the general starting point for any Godunov-type finite-volume scheme, and the careful reader will notice that (8) is a special case of (19). To make a complete numerical scheme we must now define how to compute the integrals in (18) and (19). This can generally done through a three-step algorithm called *reconstruct–evolve–average* (REA) due to Godunov:

(1) Starting from known cell-averages u_i^n in grid cell $[x_{i-1/2}, x_{i+1/2}]$ at time $t = t_n$, we **reconstruct** a piecewise polynomial function $\hat{u}(x,t_n)$ defined for all points x. The simplest possible choice is to use a piecewise constant approximation such that

$$\hat{u}(x,t_n) = u_i^n, \quad \forall x \in [x_{i-1/2}, x_{i+1/2}].$$

This will generally result in a method that is formally *first order*. To obtain a method of higher order, we use a piecewise polynomial interpolant $p_i(x)$ such that

$$\hat{u}(x,t_n) = \sum_i p_i(x)\chi_i(x),$$

where $\chi_i(x)$ is the characteristic function of the ith grid cell $[x_{i-1/2}, x_{i+1/2}]$.

(2) Then we **evolve** the hyperbolic equation (1) exactly (or approximately) with initial data $\hat{u}(x,t_n)$ to obtain a function $\hat{u}(x,t_n+\Delta t)$ a time Δt later.

(3) Finally, we **average** the function $\hat{u}(x,t_n+\Delta t)$ over an interval I as in the definition of a sliding average (18).

The averaging step generally leaves us with two basically different choices, leading to two classes of methods. Choosing $x = x_i$ in (18) gives what is referred to as *upwind* methods and choosing $x = x_{i+1/2}$ gives *central* (difference) methods, see Figure 3. To see the fundamental difference between these two classes of methods we look at the temporal integrals in (19). In the upwind class, the integral of $f(u(\cdot,t))$ is taken over points $x_{i\pm1/2}$, where the piecewise polynomial reconstruction $\hat{u}(x,t_n)$ is discontinuous. This means that one cannot apply a standard integration rule in combination with a standard extrapolation. Instead, one must first resolve the local wave-structure arising due to the discontinuity. This amounts to solving

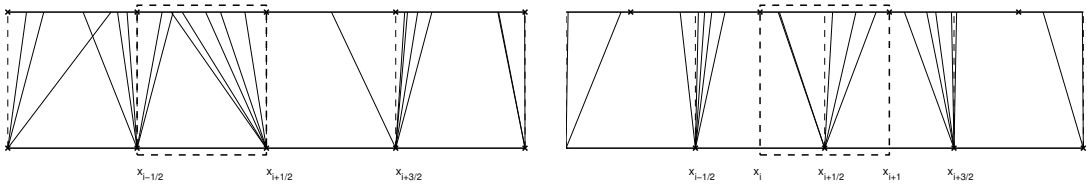

FIGURE 3. Computation of sliding averages for upwind schemes (left) and central schemes (right).

a so-called *Riemann problem*. We will come back to this briefly below. For the central methods, the sliding average is computed over a *staggered* grid cell $[x_i, x_{i+1}]$, which means that the flux integral is evaluated at the points x_i and x_{i+1}, where the initial data $\hat{u}(x, t_n)$ is smooth. If the discretisation parameters satisfy a CFL condition, which states that λ times the maximum local wavespeed is less than one half, the solution $\hat{u}(x, t)$ will remain smooth at these points for $t \in [t_n, t_n + \Delta t]$. The flux integral can thus be computed using some standard integration scheme in combination with a straightforward extrapolation according to (1).

6.1. **High-resolution central schemes.** The classical example of a central difference scheme is the Lax–Friedrichs scheme, as given in (11). The Lax–Friedrichs scheme has a *staggered* version, which can be derived within the Godunov-framework introduced in Section 6 if we assume a piecewise constant reconstruction and use a one-sided quadrature rule for the flux integrals in (19)

$$u_{i+1/2}^{n+1} = \tfrac{1}{2}\big(u_i^n + u_{i+1}^n\big) - \lambda\big[f(u_{i+1}^n) - f(u_i^n)\big],$$

The scheme is stable under the CFL restriction $(\Delta t/\Delta x)\max_u |f'(u)| \le 1/2$. Notice that this scheme can be converted to a *nonstaggered* scheme by averaging the staggered cell-averages over the original grid

$$u_i^{n+1} = \tfrac{1}{2}\big(u_{i-1/2}^{n+1} + u_{i+1/2}^{n+1}\big) = \tfrac{1}{4}\big(u_{i-1}^n + 2u_i^n + u_{i+1}^n\big) - \tfrac{1}{2}\lambda\big[f(u_{i+1}^n) - f(u_{i-1}^n)\big],$$

which is almost on the same form as the Lax–Friedrichs given in (11).

As an example of high-resolution schemes, we will now derive the second-order extension of the staggered Lax–Friedrichs scheme as introduced by Nessyahu–Tadmor [21]. This scheme, henceforth referred to as NT, is the simplest possible example of high-resolution central schemes.

For simplicity, we first consider the scalar case. Assume a grid with uniform cell size Δx. Let u_i^n approximate the cell-average over the ith cell $[x_{i-1/2}, x_{i+1/2}]$ at time $t^n = n\Delta t$ and $u_{i+1/2}^{n+1}$ the cell-average over the staggered cell $[x_i, x_{i+1}]$ at time $t^{n+1} = (n+1)\Delta t$. Since we seek a second-order method, the scheme starts with a piecewise linear reconstruction,

$$\hat{u}_i(x, t_n) = u_i^n + (x - x_i)s_i, \qquad x \in [x_{i-1/2}, x_{i+1/2}].$$

If we now insert this into (19) and evaluate the sliding average over the staggered grid cell, we obtain

$$\begin{aligned} u_{i+1/2}^{n+1} &= \int_{x_i}^{x_{i+1}} \hat{u}(x,t)\,dx - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \big[f(\hat{u}(x_{i+1},t)) - f(\hat{u}(x_i,t))\big]\,dt \\ &= \frac{1}{2}\big(u_i^n + u_{i+1}^n\big) + \frac{\Delta x}{8}\big(s_i - s_{i+1}\big) - \frac{1}{\Delta x}\int_{t_n}^{t_{n+1}} \big[f(\hat{u}(x_{i+1},t)) - f(\hat{u}(x_i,t))\big]\,dt. \end{aligned}$$

It turns out that it is sufficient to approximate the flux integrals by the midpoint rule to obtain second-order accuracy; that is, we set

$$\int_{t_n}^{t_{n+1}} f(\hat{u}(x_i,t))\,dt \approx \Delta t f(\hat{u}(x_i, t_n + \tfrac{1}{2}\Delta t)),$$

and similarly at point $x = x_{i+1}$. To complete the scheme, we need to determine how to compute the *point-values* $\hat{u}(x_i, t_{n+1/2})$ and $\hat{u}(x_{i+1}, t_{n+1/2})$. If we now assume that the discretisation satisfies a CFL condition $(\Delta t/\Delta t)\max|f'(u)| \leq 1/2$, the solution $\hat{u}(\cdot,t)$ will be continuous at the midpoints. Thus, we can use an extrapolation of (1) in time using a Taylor series

$$\hat{u}(x_i, t_{n+1/2}) \approx \hat{u}(x_i,t_n) - \frac{\Delta t}{2} f(\hat{u}(x_i,t_n))_x \approx u_i^n - \frac{\Delta t}{2}\sigma_i.$$

The flux gradient σ_i can either be computed as $f'(u_i^n)s_i$ or as the slope from a piecewise linear reconstruction of the fluxes of the cell averages. (The careful reader will have noticed that for a piecewise linear reconstruction, the cell averages u_i^n coincide with the point values $\hat{u}(x_i,t_n)$). This is almost the full story of the scheme. The only delicate point we have not touched is how to compute the slopes s_i in the piecewise linear reconstructions. A natural candidate is, of course, to use discrete differences, either either one-sided or central differences. This means that the slopes s_i could be given by any of the formulas

$$s_i^- = u_i^n - u_{i-1}^n, \qquad s_i^+ = u_{i+1}^n - u_i^n, \qquad s_i^c = \frac{1}{2}\big(u_{i+1}^n - u_{i-1}^n\big).$$

Whereas the two one-sided differences are first order approximations for smooth data, the central difference is second order and would generally be the preferred choice. However, one can show that all three choices lead to schemes that are formally second-order accurate on smooth solutions of (1). For discontinuous solutions, on the other hand, using any of the three approximations may lead to the formation of unphysical oscillations that spread out from a discontinuity, as seen in Examples 1 and 2. For scalar equations, the corresponding schemes will violate two fundamental properties of the physical solution: boundedness in L^∞ and bounded variation. To illuminate this point, let us consider the following set of cell averages

$$u_i^n = \begin{cases} 1, & i \leq k, \\ 0, & i > k. \end{cases}$$

for which we have

$$s_k^- = 0, \qquad s_k^+ = -1, \qquad s_k^c = -\tfrac{1}{2}.$$

Obviously, a new maximum will be introduced in $\hat{u}(x,t_n)$ for the two candidate slopes s_k^+ and s_k^c. Similarly, if the function u_i^n is reversed from a backward to a forward step, both s_k^- and s_k^c will introduce new extrema. The formation of new extrema, and the resulting creation of unphysical oscillations, can of course be completely avoided if we use a piecewise constant approximation, but then the formal order of the scheme would be reduced to first order. Altogether, this suggests that we should try to put some more intelligence into the scheme and use the local behaviour of the cell averages to determine how to compute s_i. This "intelligence" comes in the form of a nonlinear function called a *limiter*; that is,

$$s_i = \Phi\big(u_i^n - u_{i-1}^n, u_{i+1}^n - u_i^n\big), \qquad \Phi(a,b) = \phi\Big(\frac{b}{a}\Big)a. \tag{20}$$

This limiter has much of the same purpose as the flux-limiter $\theta(u^n; i)$ introduced at the end of Section 3.

Under certain restrictions on the function ϕ, one can show that the resulting scheme has diminishing total variation; that is,

$$TV(u_{i+1/2}^{n+1}) = \sum_i |u_{i+1/2}^{n+1} - u_{i-1/2}^{n+1}| \le TV(u_i^n) = \sum_i |u_i^n - u_{i-1}^n|.$$

See [17] for a more detailed discussion of the use of limiters for the Nessyahu–Tadmor scheme. A robust example of a limiter is the minmod limiter

$$\operatorname{minmod}(a,b) = \begin{cases} a, & \text{if } |a| < |b| \text{ and } ab > 0 \\ b, & \text{if } |b| < |a| \text{ and } ab > 0 \\ 0, & \text{if } ab \le 0. \end{cases}$$

Summing up, we have derived a predictor–corrector scheme of the form

$$\begin{aligned} u_i^{n+1/2} &= u_i^n - \frac{\lambda}{2}\sigma_i, \\ u_{i+1/2}^{n+1} &= \frac{1}{2}\big(u_i^n + u_{i+1}^n\big) - \lambda\big(g_{i+1}^n - g_i^n\big), \\ g_i^n &= f(u_i^{n+1/2}) + \frac{1}{8\lambda}s_i, \\ s_i &= \Phi\big(u_i^n - u_{i-1}^n, u_{i+1}^n - u_i^n\big) \\ \sigma_i &= \Phi\big(f(u_i^n) - f(u_{i-1}^n), f(u_{i+1}^n) - f(u_i^n)\big) \end{aligned} \tag{21}$$

The scheme is formally second order. Moreover, under appropriate assumptions on the time-step and the limiter function Φ one can prove that this scheme gives solutions that are bounded by their initial data in L^∞ norm and have diminishing total variation. Thus, unlike the classical second-order schemes, the NT scheme mimics the properties of the exact scalar solution.

The major advantage of the NT scheme is that it is both compact and simple to implement, particularly since it does not require the use any characteristic information or solution of local Riemann problems (see Section 6.2). The only requirement is an estimate of the maximum wavespeed needed to impose a CFL restriction on the time step.

Example 3. *Let us now apply the NT scheme to the linear advection and Burgers' equation as considered in Examples 1 and 2. We use two different limiter functions, the dissipative minmod limiter and the compressive superbee limiter. Figure 4 shows the approximate solutions computed with the same parameters as in Figures 1 and 2, except for the time-step which is now $\Delta t = 0.475\Delta x$. The improvement in the resolution is obvious. On the other hand, we see that there is some difference in the two limiter functions. The dissipative minmod limiter always chooses the lesser slopes and thus behaves more like a first-order scheme. The compressive superbee limiter picks steeper slopes and has a tendency of overcompressing smooth linear waves, as observed for the smooth cosine profile.*

If higher accuracy is wanted, one must use a spatial reconstruction of higher order and a more accurate temporal extrapolation in terms of more predictor steps like in higher-order Runge–Kutta methods, see [20, 1, 2, 15]. Similarly, semi-discrete nonstaggered schemes have been developed, for which only the spatial derivatives are discretised, leading to a set of ordinary differential equations that can be integrated by an ODE solver [11, 9, 10].

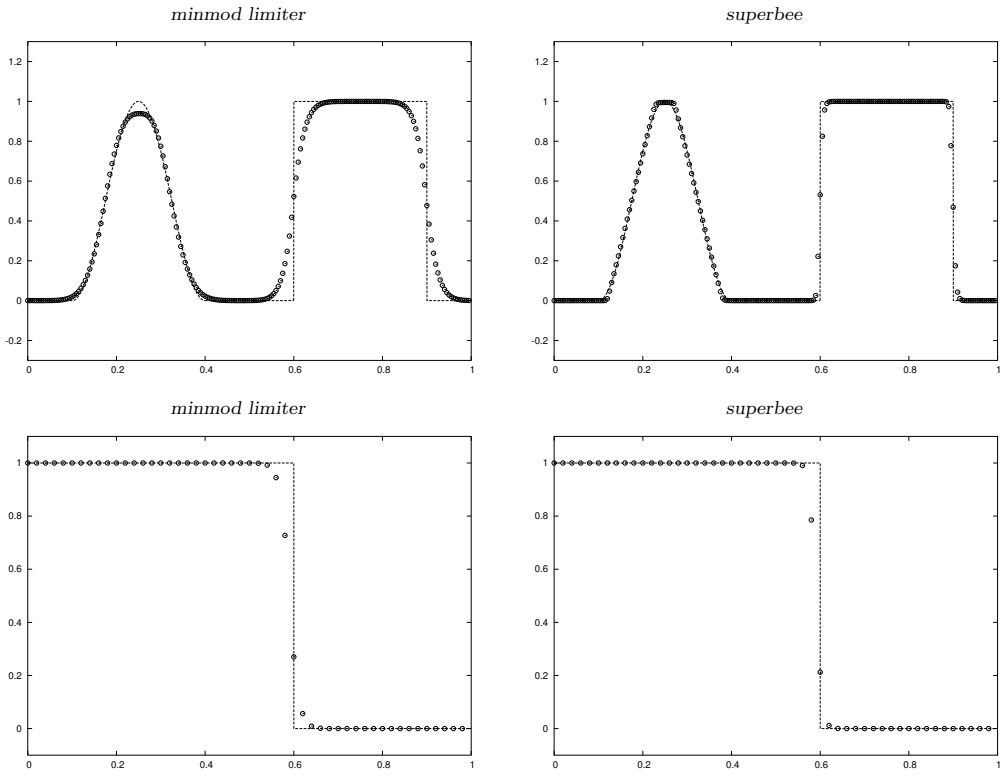

FIGURE 4. Approximate solutions of the linear advection and Burgers' equation computed by the NT scheme with two different limiters.

There are different ways to extend central schemes to systems of conservation laws. The simplest method is to apply the scheme directly to each component of the vector of unknowns [21]. This greatly simplifies the implementation of central schemes and is possible since the schemes do not use the eigenstructure of the underlying system.

To derive high-resolution schemes for conservation laws in multidimensions we can apply similar ideas. In two spatial dimensions, the conservation law reads

$$u_t + f(u)_x + g(u)_y = 0, \qquad u(x,y,0) = u_0(x,y). \tag{22}$$

As in one dimension, we introduce the sliding average

$$\bar{u}(x,y,t) = \frac{1}{\Delta x \Delta y} \int_{I(x)} \int_{J(y)} u(\xi,\eta,t)\, d\xi d\eta,$$

and integrate (22) over the domain $I(x) \times J(y) \times [t, t+\Delta t]$ to derive an evolution equation for the sliding average

$$\begin{aligned}\bar{u}(x,t+\Delta t) =& \bar{u}(x,t) \\ &- \frac{1}{\Delta x \Delta y} \int_t^{t+\Delta t} \int_{J(y)} \Big[f\big(u(x+\tfrac{1}{2}\Delta x, y, s)\big) - f\big(u(x-\tfrac{1}{2}\Delta x, y, s)\big) \Big]\, dyds \\ &- \frac{1}{\Delta x \Delta y} \int_t^{t+\Delta t} \int_{I(x)} \Big[g\big(u(x, y+\tfrac{1}{2}\Delta y, s)\big) - g\big(u(x, y-\tfrac{1}{2}\Delta y, s)\big) \Big]\, dxds.\end{aligned}$$

Secondly, we make a piecewise linear reconstruction in each spatial direction,

$$\hat{u}_{ij}(x,y,t_n) = u^n_{ij} + (x-x_i)s^x_{ij} + (y-y_j)s^y_{ij}, \qquad (x,y) \in [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}].$$

We now evaluate the sliding average over the staggered grid cell, defined analogously as for the one-dimensional case, and use the midpoint rule to approximate the flux integrals. Altogether this gives the two-dimensional version of the Nessyahu–Tadmor scheme [6],

$$\begin{aligned}
u^{n+1}_{i+1/2,j+1/2} =& \frac{1}{4}\big(u^n_{ij} + u^n_{i+1,j} + u^n_{i+1,j+1} + u^n_{i,j+1}\big) \\
&+ \frac{1}{16}\big(s^x_{ij} + s^x_{i,j+1} - s^x_{i+1,j} - s^x_{i+1,j+1}\big) \\
&+ \frac{1}{16}\big(s^y_{ij} - s^y_{i,j+1} + s^y_{i+1,j} - s^y_{i+1,j+1}\big) \\
&- \frac{\lambda}{2}\Big[f(u^{n+1/2}_{i+1,j}) + f(u^{n+1/2}_{i+1,j+1})\Big] + \frac{\lambda}{2}\Big[f(u^{n+1/2}_{i,j}) + f(u^{n+1/2}_{i,j+1})\Big] \\
&- \frac{\mu}{2}\Big[g(u^{n+1/2}_{i,j+1}) + g(u^{n+1/2}_{i+1,j+1})\Big] + \frac{\mu}{2}\Big[g(u^{n+1/2}_{i,j}) + g(u^{n+1/2}_{i+1,j})\Big], \\
u^{n+1/2}_{ij} =& u^n_{ij} - \frac{\lambda}{2}\sigma^x_{ij} - \frac{\mu}{2}\sigma^y_{ij},
\end{aligned} \tag{23}$$

where $\lambda = \Delta t/\Delta x$, $\mu = \Delta t/\Delta y$. As for its one-dimensional counterpart, the scheme is compact and easy to implement, can be applied to systems in a componentwise fashion, and has fairly good accuracy.

High-resolution central schemes have seen a rapid development since the late 90ties and are established as simple, but versatile schemes for integrating conservation laws in several dimensions. We refer the reader to [22] for a complete overview of extensions to higher order, unstructured grids, semi-discrete versions, and applications proving the versatility of the schemes.

6.2. **High-resolution upwind schemes.** To derive upwind schemes, we return to the sliding average in (18) Let us for simplicity assume that the reconstructed function $\hat{u}(x,t_n)$ is piecewise constant. To evolve the solution, we see from Figure 3 that in order to compute the integral of the flux function over the cell boundaries, we must solve a series of simple initial-value problems of the form,

$$u_t + f(u)_x = 0, \qquad u(x,0) = \begin{cases} u_L, & x < 0, \\ u_R, & x > 0. \end{cases}$$

This is commonly referred to as a Riemann problem, which has a self-similar solution of the form $u(x,t) = v(x/t; u_L, u_R)$ and consists of a set of constant states separated by simple waves (rarefaction, shocks and contacts). Since a hyperbolic equation has finite speed of propagation, the *global* solution $\hat{u}(x,t)$ for sufficiently small t can be constructed by piecing together the local Riemann solutions. In general it can be quite complicated to solve this Riemann problem, at least for systems of conservation laws. However, to compute the flux integrals in (19) we only need the solution of the Riemann problem along the ray $x/t = 0$, where the solution is constant $u(0,t) = v(0; u_L, u_R)$. Thus, the general form of the upwind Godunov-methods reads

$$u^{n+1}_i = u^n_i - \lambda\Big[f\big(v(0; u^n_{i-1}, u^n_i)\big) - f\big(v(0; u^n_i, u^n_{i+1})\big)\Big]. \tag{24}$$

A specific scheme is obtained by devising a method to compute the (approximate) solution of the local Riemann problems.

Example 4. *Let us consider a convex flux function that satisfies $f''(u) > 0$ (the case $f''(u) < 0$ is similar). In this case the Riemann solution consists of either a single rarefaction wave or a single shock. If $f'(u)$ is either strictly positive or strictly negative, the single wave will move to one side only, and the Godunov scheme simplifies to the upwind scheme* (10)*. If not, the solution must consist of a rarefaction wave moving in both the positive and negative direction. Inside the rarefaction wave there is a single sonic point u_s where $f'(u_s)$ is zero, and the wave is therefore called a transonic rarefaction wave. Summing up our observations, the Godunov flux reads*

$$F^n_{i+1/2} = \begin{cases} f(u^n_i), & s_{i+1/2} > 0 \text{ and } u^n_i > u_s \\ f(u^n_{i+1}), & s_{i+1/2} < 0 \text{ and } u^n_{i+1} < u_s \\ f(u_s), & u^n_i < u_s < u^n_{i+1}. \end{cases}$$

Here $s_{i+1/2} = [f(u^n_{i+1}) - f(u^n_i)]/(u^n_{i+1} - u^n_i)$ is the Rankine–Hugoniot speed associated with the jump. A similar formula holds for the case when $f''(u) < 0$.

The Godunov flux derived in the above example can be written in a more compact form, which is also valid for an arbitrary nonconvex flux function $f(u)$

$$F^n_{i+1/2} = \begin{cases} \min_{u\in[u^n_i,u^n_{i+1}]} f(u), & u^n_i \le u^n_{i+1}, \\ \max_{u\in[u^n_{i+1},u^n_i]} f(u), & u^n_i \ge u^n_{i+1}. \end{cases} \tag{25}$$

If $f(u)$ is nonconvex, the flux may have several sonic points, one at each of its local critical points.

Working with the exact Godunov flux in an actual implementation is a bit cumbersome since the formula requires the computation of the minimum or maximum of $f(u)$ over an interval. It is therefore customary to replace the formula (25) with another formula based upon an approximation to the Riemann problem. This approach is also much easier to generalise to systems of conservation laws.

Let us first assume that the solution is a continuous wave (i.e., a rarefaction wave). This gives the Engquist–Osher scheme, which is a natural extension of the upwind scheme to nonconvex flux functions. The Engquist–Osher flux function reads

$$F^n_{i+1/2} = f(0) + \int_0^{u^n_i} \max(f'(v), 0)\, dv + \int_0^{u^n_{i+1}} \min(f'(v), 0)\, dv. \tag{26}$$

Alternatively, we can approximate the Riemann problem by a single shock. Then the flux can be written as

$$F^n_{i+1/2} = f(u^n_i) + s^-_{i+1/2}(u^n_{i+1} - u^n_i),$$

or alternatively as

$$F^n_{i+1/2} = f(u^n_{i+1}) - s^+_{i+1/2}(u^n_{i+1} - u^n_i).$$

Here $s^+ = \max(s, 0)$ and $s^- = \min(s, 0)$. By averaging the equivalent expressions we obtain the numerical flux

$$F^n_{i+1/2} = \tfrac{1}{2}\big[f(u^n_i) + f(u^n_{i+1}) - |s_{i+1/2}|(u^n_{i+1/2} - u^n_i)\big].$$

This can be interpreted as a central flux approximation plus a viscous correction with coefficient $|s_{i+1/2}|$. The formula can quite easily be extended to systems of equations and gives

what is commonly referred to as the Roe linearisation of the Riemann problem. For transonic waves the coefficient $s_{i+1/2}$ may vanish or be close to zero and the added dissipation is insufficient to stabilise the computation. It is therefore customary to add extra dissipation in the form of an entropy fix. For more details, consult for instance [14].

High-resolution versions of the upwind schemes can be obtained by using a higher-order reconstruction of the cell averages. This is beyond the scope of the exposition. The interested reader can find details in the books by Godlewski and Raviart [4], Holden and Risebro [5], Kröner [7], LeVeque [13, 14], and Toro [23].

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