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Department Mathematics and Statistics
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Winter Semester 2017/2018

Exercises for
Theory and Numerics of Partial Differential Equations

http://www.math.uni-konstanz.de/numerik/personen/mechelli/teaching.html

Exercise Sheet 12

Submission[1]: 9th February at 15:00

Exercise 1. (Theory, 10 points)
Let $T > 0$ be a final time and V, H Hilbert spaces such that $V \hookrightarrow H = H' \hookrightarrow V'$ is a *Gelfand triple*[2]. Let further $A \in L(V, V')$ and $f \in L^\infty(0, T, V')$ as well as $y_0 \in H$. If there exist $\alpha > 0$ and $\beta \geq 0$ such that

$$\langle A\varphi, \varphi\rangle_{V' \times V} \;\geq\; \alpha\|\varphi\|_V^2 - \beta\|\varphi\|_H^2 \quad \forall \varphi \in V$$

then the following is known as an *abstract parabolic evolution equation*:

$$\begin{aligned} y_t(t) + Ay(t) &= f(t) && \text{in } V' \text{ for almost all } t \in (0, T) \\ y(0) &= y_0 && \text{in } H \end{aligned} \tag{PB}$$

It can be shown that a unique solution of (PB) exists with

$$y \in W(0, T) := L^2(0, T; V) \cap H^1(0, T; V') \hookrightarrow C([0, T]; H),$$

where the last embedding is a well-known property of the space $W(0, T)$. Your task is to prove the following Theorem which estimates the energy of the solution y against the energy of the initial data f and y_0:

Theorem 1. For all solutions $y \in W(0, T)$ of (PB), it holds

$$\|y(T)\|_H^2 + \alpha\|y\|_{L^2(0,T;V)}^2 \;\leq\; e^{2\beta T}\left(\|y_0\|_H^2 + \tfrac{1}{\alpha}\|f\|_{L^2(0,T;V')}^2\right).$$

Prove it by the following steps:

(1) Derive an estimate for the term $\frac{d}{dt}\|y(t)\|_H^2$. In order to do this, you may use the fact that $\frac{1}{2}\frac{d}{dt}\|y(t)\|_H^2 = \langle y_t(t), y(t)\rangle_{V' \times V}$. Young's inequality is also helpful.

(2) Use Gronwall's Lemma[3] to derive an estimate for the term $\|y(t)\|_H^2$ based on your estimate from step (1).

(3) Integrate your estimate from step (1) over $(0, T)$ and use your estimate from step (2) to complete the proof.

Exercise 2. (Programming, 10 points)
Using Matlab *PDE Toolbox*, solve the following Parabolic Problem:

$$\begin{cases} y_t(t, x) - \Delta y(t, x) = f(x) \text{ for all } x \in \Omega\ ,\ t \in (0, 1] \\ \eta \frac{\partial y(t,x)}{\partial n} + \alpha y(t, x) = 0 \quad \text{for all } x \in \partial\Omega\ ,\ t \in (0, 1] \\ y(0, x) = y_0(x) \end{cases} \tag{1}$$

which depends on $\alpha, \eta \in \mathbb{R}$, $\eta \neq 0$ and where $f : \Omega \to \mathbb{R}$ is a continous function and $\Omega \subset \mathbb{R}^2$ is given by the interior of the blue line in Figure 1, that depends on the parameter $a > 0$ in $\mathbb{R}$. As in the previous Sheet, declare a as global parameter in your main script and make it available in each function.

[1]Please bring the Theory Exercises to our office in Z919. The Programming Exercises (including the report) have to be sent by email to `luca.mechelli@uni-konstanz.de` (group of Tuesday) and to `hai.nguyen-pham@uni-konstanz.de` (group of Wednesday) **before** the submission's deadline.

[2]This means that V is densely embedded in H and H' is densely embedded in V'. H is identified with H' by the Riesz isomorphism.

[3]See `https://www.math.uni-bielefeld.de/~rkruse/files/gronwall.pdf`

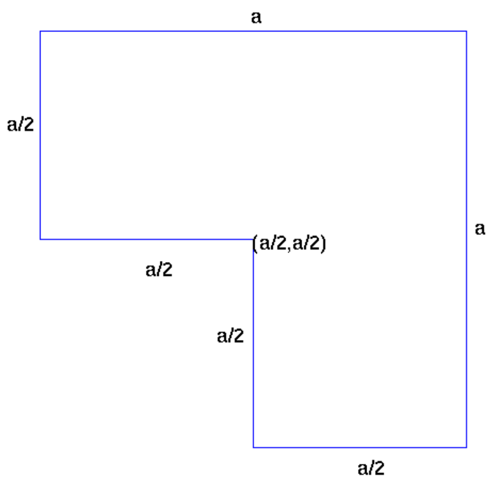

Figure 1: Domain Ω

In order to solve the problem follow these steps:

1. **Geometry Implementation:** Write a function `geometryFunction.m` to describe the geometry of Ω by using a suitable analytical boundary representation. Especially focus on the various ways that this function will be called by the PDE toolbox (0,1,2 inputs, bs scalar or a vector,...) Then use the command `pdegplot('geometryFunction')` to test your results.

The following two points should be solved in a script. **Do not use point-and-click for these!**

2. **PDE specification:** Specify the PDE coefficients in (1) and generate a mesh with maximum element size 0.05. Visualize the mesh.

3. **PDE solving:** Solve the problem for different choices of the parameters η and α[4] and for the following choices of $f(x)$ and $y_0(x)$:

   (a) $f(x) = 0$, $y_0(x) = x_1 + x_2$ with $x = (x_1, x_2) \in \Omega$;

   (b) $f(x) = a$, $y_0(x) = x_1 + x_2$;

   (c) $f(x) = \frac{1}{4}a^2 - (x_1 - \frac{1}{2}a)^2 - (x_2 - \frac{1}{2}a)^2$, $y_0(x) = a$;

   In particular, try the choice $(\eta, \alpha) = (1, 10^7)$, which kind of boundary condition does it imitate? Write a function `solve_parabolic_problem`, which solves the problem by assembling the Finite Element matrices with the command `assembleFEMatrices` and use *implicit Euler* scheme in time to get the solution[5] $y(t_s, \cdot)$ at every time step t_s.[6] The function should also plot the time evolution of the solution $y(t, x)$ with $t \in [0, 1]$ and $x \in \Omega$.

Write a thorough report documenting how the variation of η, α and N_t, the number of time steps used for the time discretization, affects the solution.

[4] MATLAB let you set only the α coefficient in a simple way, so we suggest to divide the boundary equation by η to obtain the new equation $\frac{\partial y(t,x)}{\partial n} + \frac{\alpha}{\eta} y(t, x) = 0$. Notice, also, that as in the previous sheet for some combination of parameter, the solution is really 'ugly'.

[5] Use the \ command to solve the linear systems.

[6] There are several ways to solve (1) in MATLAB, but, in order to let you make experience with finite element matrices, we recommend this procedure. For `assembleFEMatrices` look at this link: `https://de.mathworks.com/help/pde/ug/assemblefematrices.html`