University of Konstanz
Department Mathematics and Statistics
L. Mechelli, J. Lu, Prof. Dr. S. Volkwein

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Exercises for
Theory and Numerics of Partial Differential Equations

http://www.math.uni-konstanz.de/numerik/personen/mechelli/teaching.html

Exercise Sheet 7

Submission[1]: 21$^{\text{st}}$ December at 10:00

Exercise 1. (Theory, 10 points)
Let $y \in C^1([0,T],\mathbb{R})$ and consider the following autonomous Cauchy Problem:

$$\begin{cases} \dot{y}(t) = f(y(t)) \\ y(0) = y_0 \end{cases} \tag{1}$$

with $f : \mathbb{R} \to \mathbb{R}$ a Lipschitz continuous function.
Derive the order of consistency, and thus of convergence, of Explicit Euler, Implicit Euler, Heun[2] and Crank-Nicolson numerical schemes. (For simplicity, you can assume $y \in C^\infty([0,T],\mathbb{R})$ and $f \in C^\infty(\mathbb{R})$.)

Exercise 2. (Programming, 10 points)
Let consider a time interval $[0,T] \subset \mathbb{R}$ with $T > 0$. Let $y_i \in C^1([0,T],\mathbb{R})$ for $i = 1,\dots,N-1$ ($N \in \mathbb{N}$). Consider the following system of linear ODEs:

$$\begin{cases} \dot{y} = Ay \\ y_i(0) = \sin(\frac{i\pi}{N}) \end{cases} \tag{2}$$

where $y = (y_1,\dots,y_{N-1})$, $N \geq 20$ and $A \in \mathbb{R}^{(N-1)\times(N-1)}$ is a matrix with the following structure:

$$A_{ij} = \begin{cases} -2N^2 & \text{if } i = j \\ N^2 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Solve (2) numerically by implementing the Explicit Euler, Implicit Euler, Heun and Crank-Nicolson methods. Your codes have to be structured in the following way:

1. Each method has to be implemented separately as a function `Method_name(t,Δt,A,y0)`, where `t` is the time discretization vector representing $[0,T]$ according to the time step Δt and `y0` is the vector containing the information of y at time 0. The function `Method_name(t,Δt,A,y0)` has to give in output the numerical solution y of the system (2) as a matrix, which has as i-th row the vector containing the values at every discrete time of y_i.

2. In your main file `main_7.m`, for each method, verify the order of convergence respect to the maximum norm, plotting the decay of the error between the exact[3] and the approximated solution[4]. As last, fix a time step Δt and plot[5] at each discrete time the vector solution $y(t) = (y_1(t),\dots,y_{N-1}(t))$, computed with the method you prefer most.

Discuss in your report the order of convergence of each method, reporting and plotting the average error e_N for different values of Δt.

[1] The Theory Exercises will be collected at the begin of the lecture. The Programming Exercises (including the report) has to be sent by email to `luca.mechelli@uni-konstanz.de` (group of Tuesday) and to `hai.nguyen-pham@uni-konstanz.de` (group of Wednesday) **before** the submission's deadline.

[2] $y^{k+1} = y^k + \frac{\Delta t}{2}(f(y^k) + f(y^k + \Delta t f(y^k)))$, with $y^k = y(t^k) = y(k\Delta t)$ and $\Delta t > 0$ a discretization's time step.

[3] How does the exact solution for a system of linear ODEs look like?

[4] Hint: the order of convergence is verified for the approximation of each y_i. Since we have a linear system of ODEs, a way to verify the order of convergence is to consider the average error $e_N = (N-1)^{-1}\sum_{i=1}^{N-1}\|y_i - y_i^{exact}\|_\infty$.

[5] All the vectors $y(t)$ should be plotted in the same figure.