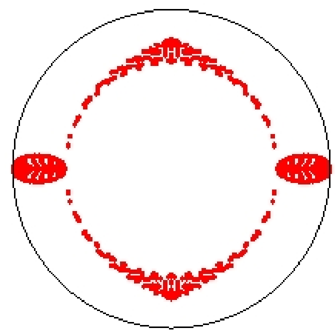
Analysis of Lattice-Boltzmann Methods

Asymptotic and Numeric Investigation of a Singularly Perturbed System

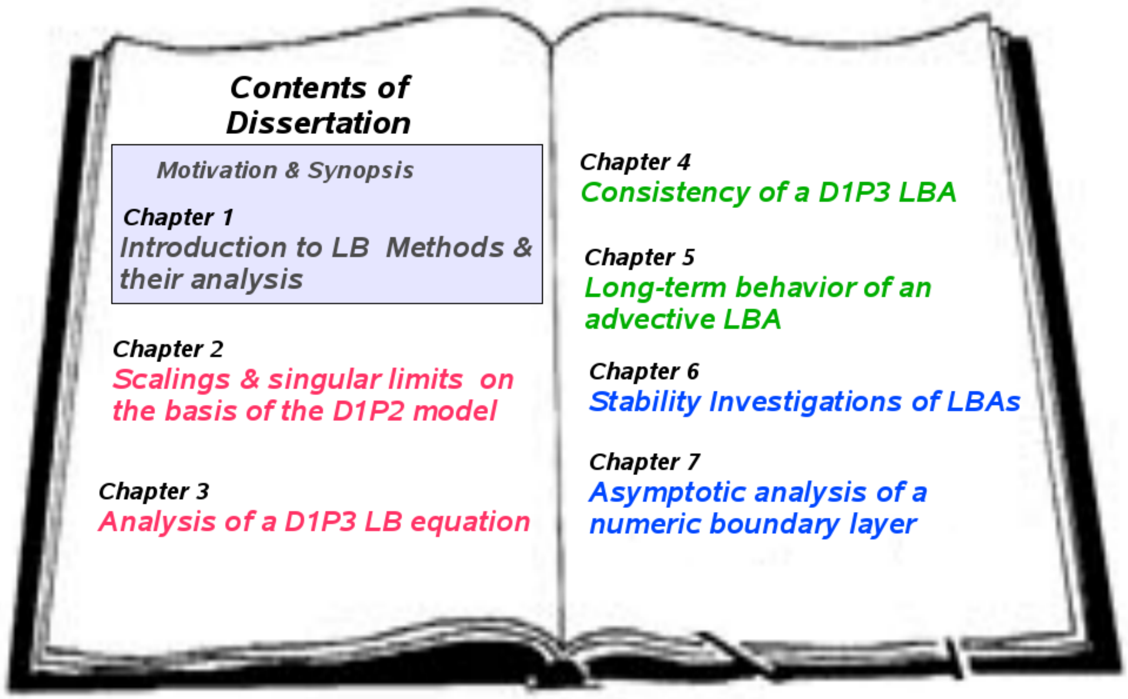


Martin Rheinländer

FB Mathematik & Statistik, AG Numerik
Universität Konstanz

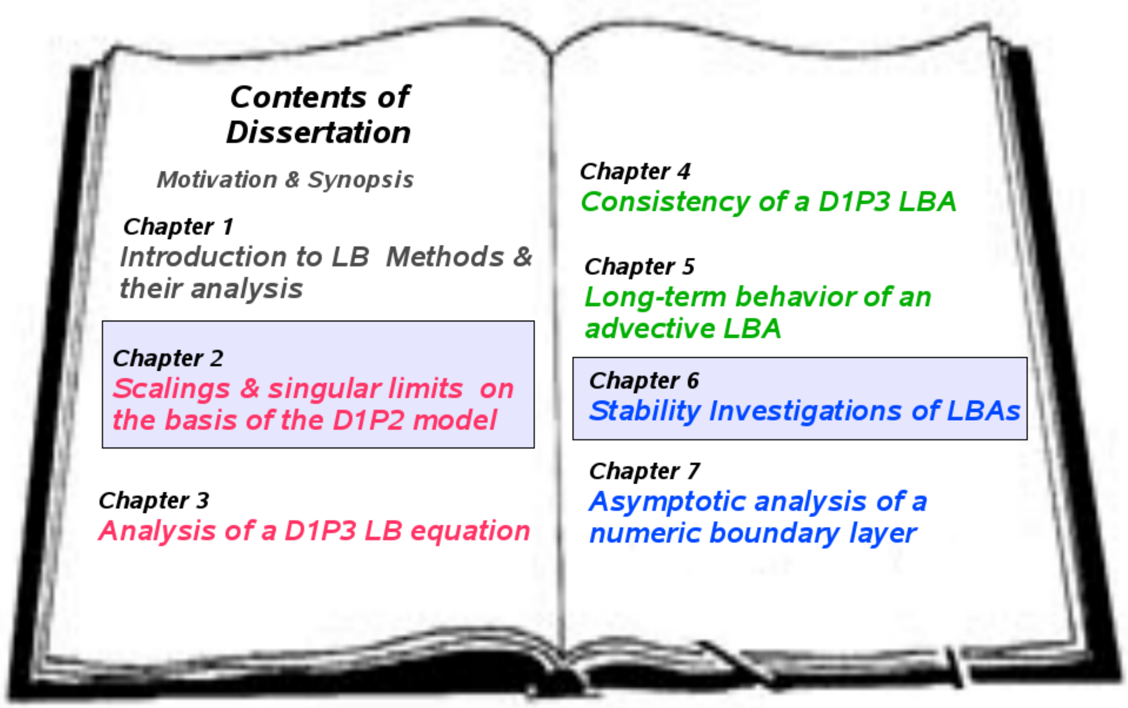
8. Mai 2007

Vortrag im Rahmen des Promotionsverfahrens



- **Part I: Introduction**
  - General concept & context of LBM
  - **Why?** Specific motivation of my work
  - **What?** Objects of analysis
    - $\longrightarrow$ Derivation of model algorithms
  - **How?** Applied methodology

Outline



- **Part I: Introduction**
  - General concept & context of LBM
  - **Why?** Specific motivation of my work
  - **What?** Objects of analysis
    → Derivation of model algorithms
  - **How?** Applied methodology

- **Part II: Results**
  - Exemplary singular limit: convergence & arising of initial layers
  - Stability & CFL condition for an LBA
  - ...

Wether in politics or mathematics ...
concrete ⇒ discrete

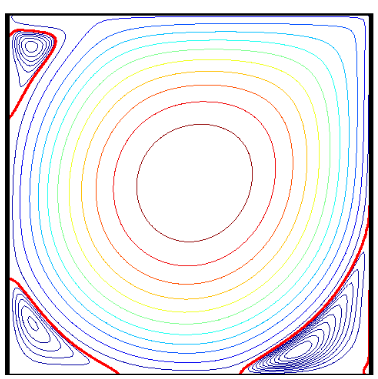
Part I – Introduction

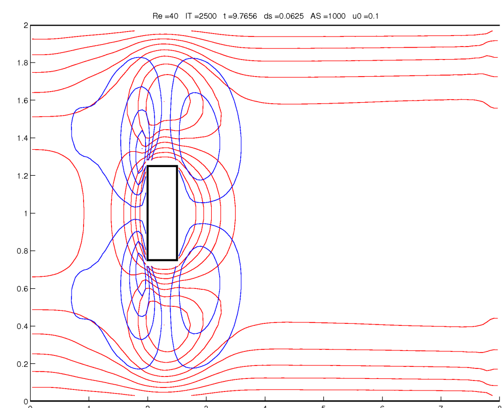
What are Lattice-Boltzmann Methods (LBM)?

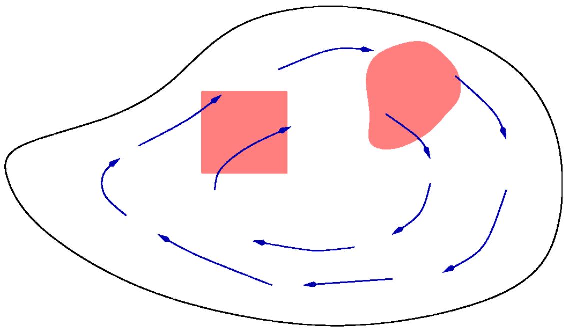
- Numeric approach for computing solutions of certain (evolutionary) PDEs.
  $\Rightarrow$ Alternative to traditional schemes: FDM, FEM and FVM.

- **Key features**:
  + *Indirect* discretization realizing a *mesoscopic* ansatz (additional variables simplify numerics).
    $\Rightarrow$ Connection to target equation is *a priori* not obvious.
  + Implemention of relatively low complexity $\longrightarrow$ well suited for parallelization.
  – Restrictions: explicit scheme, regular grids (adaptivity?), memory intensive, ...

- **Main applications:**
  – Various engineering problems with *fluid-dynamic* background.

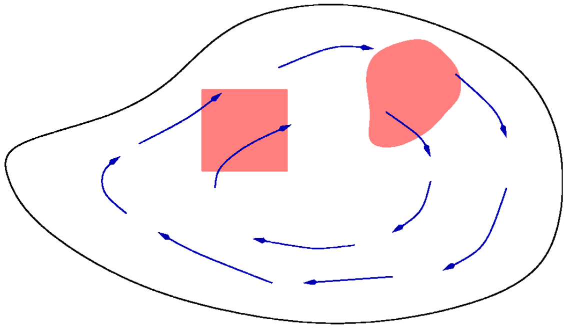


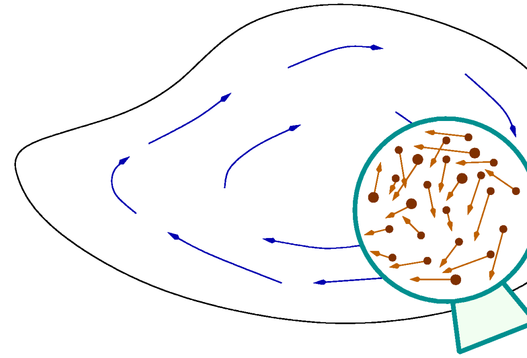




- **Macroscopic view:** continuum hypothesis

$\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0,\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p, \ldots$



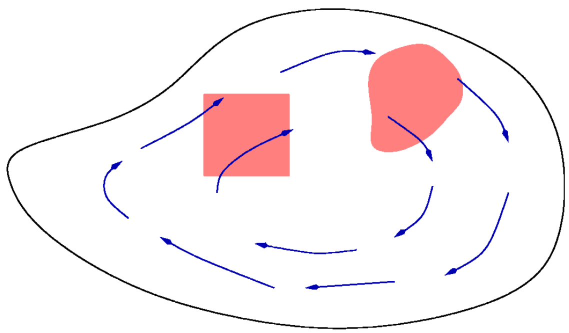


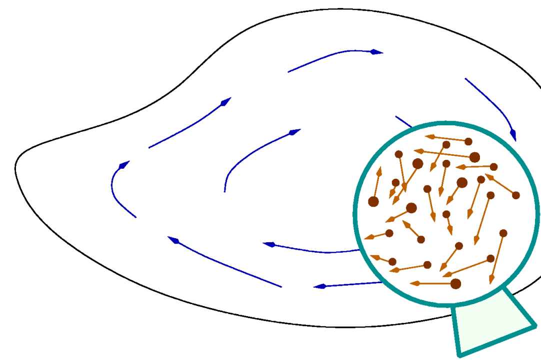
- **Macroscopic view:** continuum hypothesis
  $\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0,\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p, \ldots$
- **Microscopic view:** particle dynamics
  $\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}_i = \nabla_{\mathbf{p}_i} H,\ \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{p}_i = -\nabla_{\mathbf{x}_i} H$

- **Macroscopic view:** continuum hypothesis
  $\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0,\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p, \ldots$
- **Microscopic view:** particle dynamics
  $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i = \nabla_{\mathbf{p}_i} H,\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_i = -\nabla_{\mathbf{x}_i} H$

Observation: Complex macroscopic process $\longrightarrow$ microscopically rather simple dynamics.

Promising perspective for simulations!? $\longrightarrow$ Yes, **but** huge number of particles.





- **Macroscopic view:** continuum hypothesis

  $\partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p, \ldots$

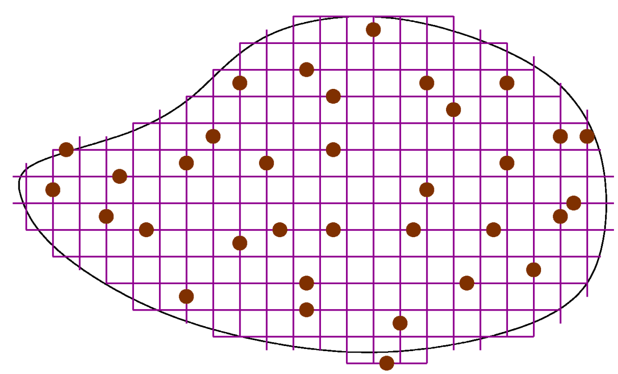
- **Microscopic view:** particle dynamics

  $\frac{\mathrm{d}}{\mathrm{d}t} \mathbf{x}_i = \nabla_{\mathbf{p}_i} H, \ \frac{\mathrm{d}}{\mathrm{d}t} \mathbf{p}_i = -\nabla_{\mathbf{x}_i} H$

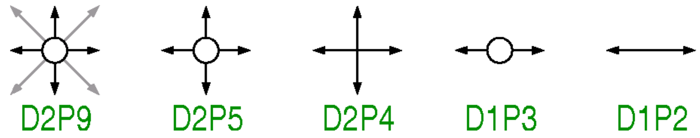
Observation: Complex macroscopic process $\longrightarrow$ microscopically rather simple dynamics.

Promising perspective for simulations!? $\longrightarrow$ Yes, **but** huge number of particles.

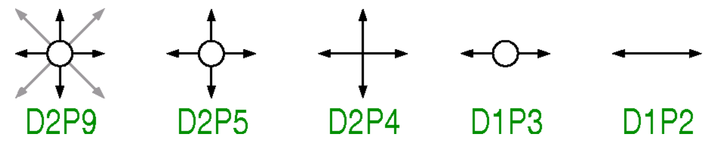
- **Mesoscopic approach:** further simplifications
  - Shrinkage of velocity space: $\mathbb{R}^n \longrightarrow \mathcal{S}$ (Finite velocity Boltzmann equations)
  - Discrete dynamical systems $\longrightarrow$ LBM



- **Discrete velocity set $\mathcal{S}$:**

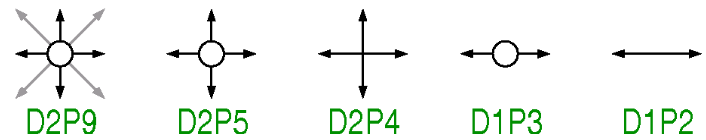


- **Discrete velocity set $\mathcal{S}$:**



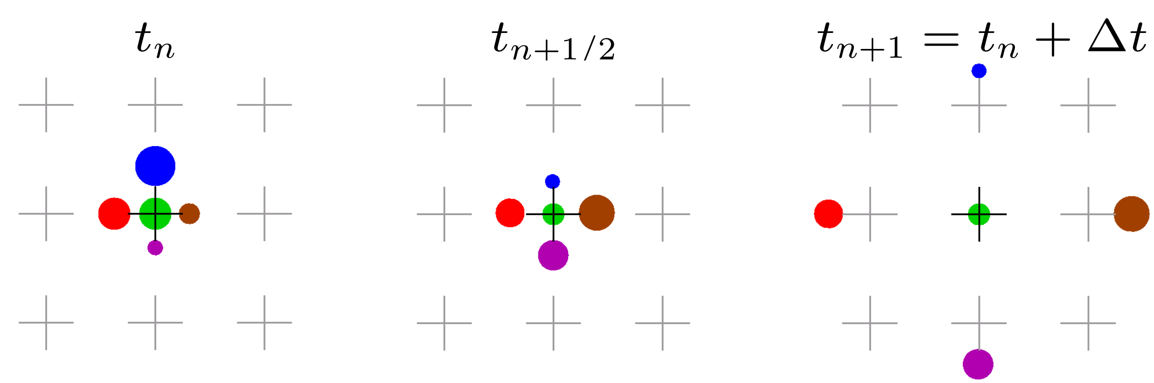
- **Primary variables:** $\mathsf{F}(t,\mathbf{x}) = [\mathsf{F}_\mathbf{s}(t,\mathbf{x})]_{\mathbf{s}\in\mathcal{S}}$ { densities of fictitious particles; grid function with $\#\mathcal{S}$ components

- **Discrete velocity set $\mathcal{S}$:**



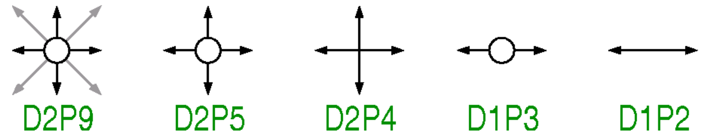
- **Primary variables:** $\mathsf{F}(t,\mathbf{x}) = [\mathsf{F}_\mathrm{s}(t,\mathbf{x})]_{\mathrm{s}\in\mathcal{S}}$ $\begin{cases} \text{densities of fictitious particles} \\ \text{grid function with } \#\mathcal{S} \text{ components} \end{cases}$

- **Evolution step:**
  1) Collision
  2) Transport



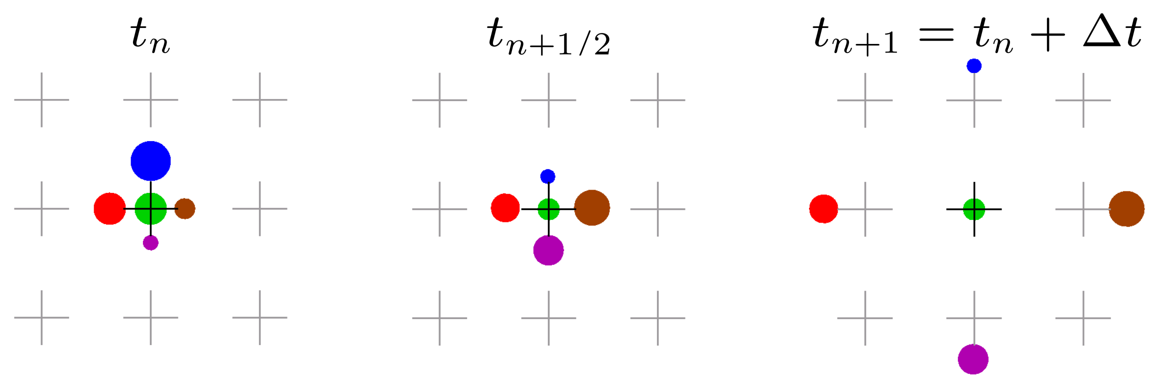
$$\mathsf{F}_\mathrm{s}(t+\Delta t, \mathbf{x}+\Delta x \mathbf{s}) = \underbrace{\mathsf{F}_\mathrm{s}(t,\mathbf{x}) + [J\mathsf{F}(t,\mathbf{x})]_\mathrm{s}}_{\text{collision product}}$$

- **Discrete velocity set $\mathcal{S}$:**



- **Primary variables:** $\mathsf{F}(t,\mathbf{x}) = [\mathsf{F}_\mathrm{s}(t,\mathbf{x})]_{\mathrm{s}\in\mathcal{S}}$ $\begin{cases} \text{densities of fictitious particles} \\ \text{grid function with } \#\mathcal{S} \text{ components} \end{cases}$

- **Evolution step:**
  1) Collision
  2) Transport



$$\mathsf{F}_\mathrm{s}(t+\Delta t, \mathbf{x}+\Delta x \mathrm{s}) = \underbrace{\mathsf{F}_\mathrm{s}(t,\mathbf{x}) + [J\mathsf{F}(t,\mathbf{x})]_\mathrm{s}}_{\text{collision product}}$$

- **Moments** approximate solution of Navier-Stokes eqn. (target eqn.):

$$R(t,\mathbf{x}) = \langle \mathsf{F}(t,\mathbf{x}), 1\rangle = \sum_{\mathrm{s}\in\mathcal{S}} \mathsf{F}_\mathrm{s}(t,\mathbf{x}), \qquad U_x(t,\mathbf{x}) = \langle \mathsf{F}(t,\mathbf{x}), \mathsf{s}_x\rangle$$

Simultaneous (coupled) limit

- Scaled finite velocity Boltzmann equation (diffusive scaling):

$$\text{FVBE}(\epsilon): \qquad \partial_t \mathrm{f} + \frac{1}{\epsilon}\mathbf{s}\cdot\nabla \mathrm{f} = \frac{1}{\epsilon^2}J_\epsilon \mathrm{f}$$

- singular limit

  $\text{FVBE}(\epsilon) \xrightarrow{\epsilon \downarrow 0}$ Navier-Stokes eqn. (target eqn.)

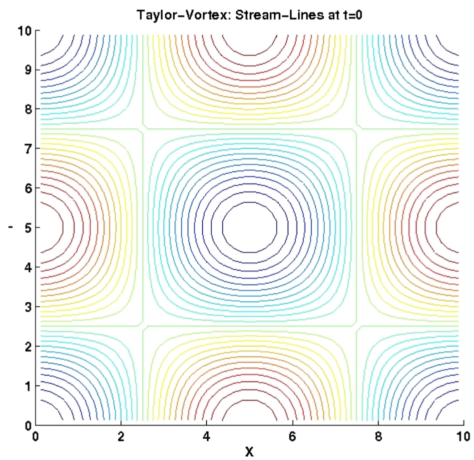
  numeric limit $\Delta t, \Delta x \downarrow 0$

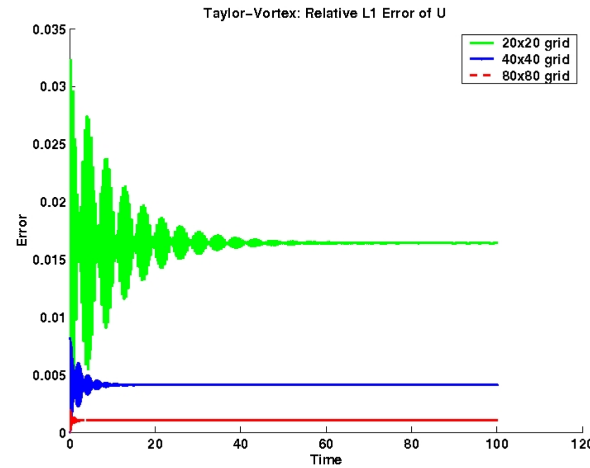
  $\left.\begin{array}{l}\Delta t = \epsilon^2 \\ \Delta x = \epsilon\end{array}\right\} \epsilon \downarrow 0$ coupled limit

  $\text{LBA}(\Delta t, \Delta x)$

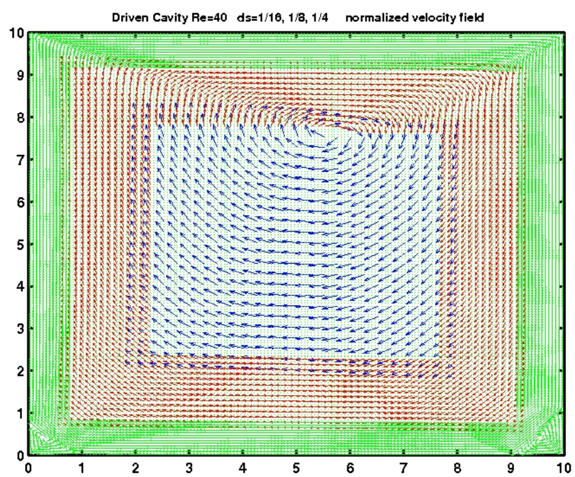
Incipient questions: consistency (traditional approach via Chapman-Enskog expansion)
convergence (requires stability)
further properties (multiple time scales, scaling, numerical layers)

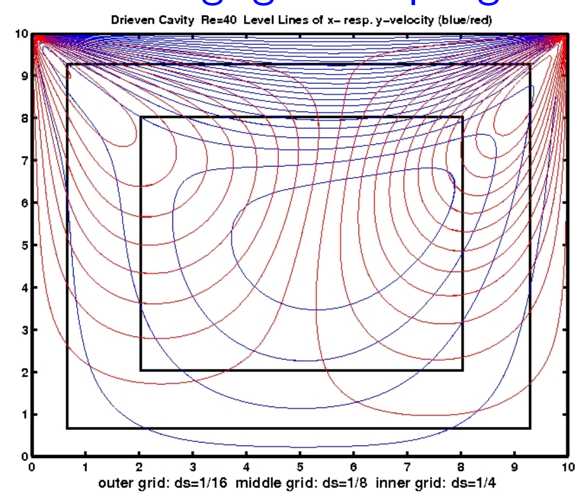
Example 1: Observation of an initial layer (simulating decaying eigenmode of Stokes operator)





Example 2: Embedding into context of further problems: e.g. grid coupling

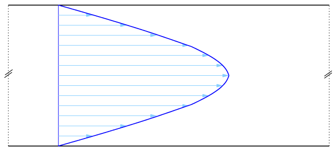




- Wanted: 1D LB algorithm $\subset$ 2D LB algorithm.
- **Reduction:** D2P9 algorithm $\longrightarrow$ D1P3 algorithm.

- Wanted: 1D LB algorithm $\subset$ 2D LB algorithm.
- **Reduction:** D2P9 algorithm $\longrightarrow$ D1P3 algorithm.
- Mimick reduction of the INS equation under translational invariance, e.g.: $\partial_x \mathbf{u} = 0$.

Parallel shear flows (*Poiseuille* flow)



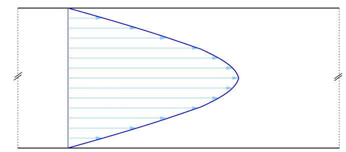
- **Fact:** If $\mathbf{u}_0(x,y) = \binom{u_0(y)}{0} \;\wedge\; \mathbf{q}(t,x,y) = \binom{q(t,y)}{0}$ then $\mathbf{u}(t,x,y) = \binom{u(t,y)}{0}$.

  Furthermore: 2D incompressible Navier-Stokes equation $\longrightarrow$ 1D diffusion equation.

$$\left.\begin{array}{c} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \nabla \mathbf{u} = -\nabla p + \mathbf{q} \end{array}\right\} \quad \longrightarrow \quad \partial_t u - \nu \partial_x^2 u = q$$

- Wanted: 1D LB algorithm $\subset$ 2D LB algorithm.
- **Reduction:** D2P9 algorithm $\longrightarrow$ D1P3 algorithm.
- Mimick reduction of the INS equation under translational invariance, e.g.: $\partial_x \mathbf{u} = 0$.

Parallel shear flows (*Poiseuille* flow)

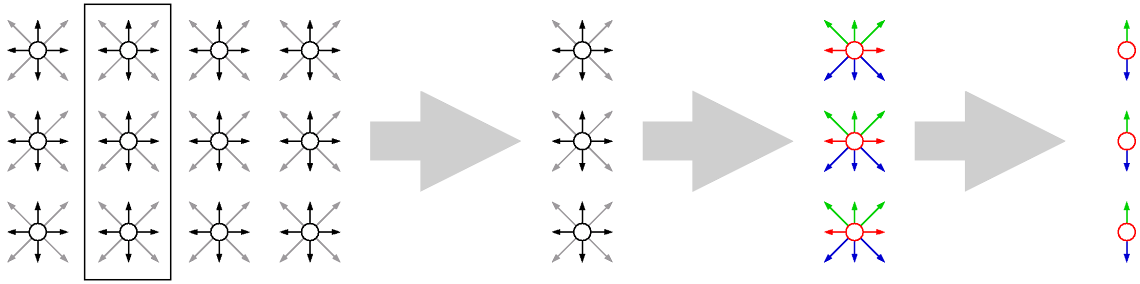


- **Fact:** If $\mathbf{u}_0(x,y) = \binom{u_0(y)}{0} \ \wedge \ \mathbf{q}(t,x,y) = \binom{q(t,y)}{0}$ then $\mathbf{u}(t,x,y) = \binom{u(t,y)}{0}$.

  Furthermore: 2D incompressible Navier-Stokes equation $\longrightarrow$ 1D diffusion equation.

$$\left.\begin{array}{c} \nabla \cdot \mathbf{u} = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \nabla \mathbf{u} = -\nabla p + \mathbf{q} \end{array}\right\} \longrightarrow \quad \partial_t u - \nu \partial_x^2 u = q$$

- Proceed analogously with LB algorithm:



  Exploit translational invariance $\rightarrow$ confine to cross-section $\rightarrow$ group 9 populations into 3 triples $\rightarrow$ define new populations.

Textbook example: $U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} a \big(U_j^n - U_{j-1}^n \big)$ $\quad \left\{ \begin{array}{c} \partial_t v + a \partial_x v = 0 \\ V_j^n := v(n\Delta t,\ j\Delta x) \end{array} \right.$

$$\underbrace{\frac{V_j^{n+1} - V_j^n}{\Delta t}}_{\partial_t v + \mathrm{O}(\Delta t)} + \ a \underbrace{\frac{V_j^n - V_{j-1}^n}{\Delta x}}_{\partial_x v + \mathrm{O}(\Delta x)} = \underbrace{R_j^n}_{\text{residue}} \qquad R_j^n = \mathrm{O}(\Delta t) + \mathrm{O}(\Delta x)$$

Vanishing residue of exact solution $\longrightarrow$ **consistency**

Textbook example: $U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} a\big(U_j^n - U_{j-1}^n\big)$ $\quad \begin{cases} \partial_t v + a\partial_x v = 0 \\ V_j^n := v(n\Delta t,\ j\Delta x) \end{cases}$

$$\underbrace{\frac{V_j^{n+1} - V_j^n}{\Delta t}}_{\partial_t v + \mathrm{O}(\Delta t)} + a \underbrace{\frac{V_j^n - V_{j-1}^n}{\Delta x}}_{\partial_x v + \mathrm{O}(\Delta x)} = \underbrace{R_j^n}_{\text{residue}} \qquad R_j^n = \mathrm{O}(\Delta t) + \mathrm{O}(\Delta x)$$

Vanishing residue of exact solution → **consistency**

LBM: $\mathsf{F}_s(n+1, j) = \mathsf{F}_s(n, j-s) + \big[J\mathsf{F}(n, j-s)\big]_s$ $\quad \begin{cases} \mathsf{F} = \big(\mathsf{F}_{-1},\ \mathsf{F}_0,\ \mathsf{F}_{+1}\big)^\top \\ v = \big(\mathsf{F}_{-1} + \mathsf{F}_0 + \mathsf{F}_{+1}\big) + Err \end{cases}$

Textbook example: $U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x} a \left(U_j^n - U_{j-1}^n \right)$ $\qquad \left\{ \begin{array}{c} \partial_t v + a \partial_x v = 0 \\ V_j^n := v(n\Delta t,\ j\Delta x) \end{array} \right.$

$$\underbrace{\frac{V_j^{n+1} - V_j^n}{\Delta t}}_{\partial_t v + \mathrm{O}(\Delta t)} + a \underbrace{\frac{V_j^n - V_{j-1}^n}{\Delta x}}_{\partial_x v + \mathrm{O}(\Delta x)} = \underbrace{R_j^n}_{\text{residue}} \qquad R_j^n = \mathrm{O}(\Delta t) + \mathrm{O}(\Delta x)$$

Vanishing residue of exact solution $\longrightarrow$ **consistency**

LBM: $\mathsf{F}_s(n+1, j) = \mathsf{F}_s(n, j-s) + \left[J\mathsf{F}(n, j-s) \right]_s$ $\qquad \left\{ \begin{array}{c} \mathsf{F} = \left(\mathsf{F}_{-1},\ \mathsf{F}_0,\ \mathsf{F}_{+1} \right)^\top \\ v = \left(\mathsf{F}_{-1} + \mathsf{F}_0 + \mathsf{F}_{+1} \right) + Err \end{array} \right.$

Consistency analysis ?

- Transformation to equivalent (moment) systems
- Expansion methods for parameter-depending problems $\longrightarrow$ generalized notion of consistency
  1) *perturbed* equations: ϵ
  2) *discretized* equations: h

$$U_j^n \approx \overbrace{u^{(0)}(nh, jh)}^{v(nh,jh)} + h u^{(1)}(nh, jh) + \ldots \qquad \text{yields unique } u^{(0)}, u^{(1)}$$

$$\mathsf{F}(n, j) \approx \mathsf{f}^{(0)}(nh, jh) + h\mathsf{f}^{(1)}(nh, jh) + \ldots \qquad \mathsf{f}^{(1)} \text{ not fully determined}$$

Part II – Results

- **Goal:** Understanding singular limits: $\left\{ \begin{array}{l} \text{Convergence} \\ \text{Arising of initial layers} \end{array} \right.$
- **Model problem:** D1P2 LB *equation* with $E\mathsf{f} = \frac{1}{2}(\mathsf{f}_1 + \mathsf{f}_2)$

$$\begin{aligned} \partial_t \mathsf{f}_1 - \epsilon^{-1} \partial_x \mathsf{f}_1 &= -\epsilon^{-2} \omega \left[\mathsf{f}_1 - E\mathsf{f} \right] = -\epsilon^{-2} \tfrac{\omega}{2} \left[\mathsf{f}_1 - \mathsf{f}_2 \right] \\ \partial_t \mathsf{f}_2 + \epsilon^{-1} \partial_x \mathsf{f}_2 &= -\epsilon^{-2} \omega \left[\mathsf{f}_2 - E\mathsf{f} \right] = -\epsilon^{-2} \tfrac{\omega}{2} \left[\mathsf{f}_2 - \mathsf{f}_1 \right] \end{aligned}$$

- **Goal:** Understanding singular limits: $\begin{cases} \text{Convergence} \\ \text{Arising of initial layers} \end{cases}$
- **Model problem:** D1P2 LB *equation* with $E\mathsf{f} = \frac{1}{2}(\mathsf{f}_1 + \mathsf{f}_2)$

$$\begin{aligned} \partial_t \mathsf{f}_1 - \epsilon^{-1}\partial_x \mathsf{f}_1 &= -\epsilon^{-2}\omega\left[\mathsf{f}_1 - E\mathsf{f}\right] = -\epsilon^{-2}\tfrac{\omega}{2}\left[\mathsf{f}_1 - \mathsf{f}_2\right] \\ \partial_t \mathsf{f}_2 + \epsilon^{-1}\partial_x \mathsf{f}_2 &= -\epsilon^{-2}\omega\left[\mathsf{f}_2 - E\mathsf{f}\right] = -\epsilon^{-2}\tfrac{\omega}{2}\left[\mathsf{f}_2 - \mathsf{f}_1\right] \end{aligned}$$

- **Reformulation:** 2×2 system $\longrightarrow$ *equivalent* scalar equation
- Mass moment: $u = \mathsf{f}_1 + \mathsf{f}_2$, 1^{st} moment (flux): $\phi = \epsilon^{-1}(\mathsf{f}_2 - \mathsf{f}_1)$
- Linear transformation $\mathsf{f}_1, \mathsf{f}_2 \leftrightarrow u, \phi$ leads to equivalent moment system:

$$\begin{aligned} \partial_t u + \partial_x \phi &= 0 & \partial_x\partial_t\phi &= -\partial_t^2 u \\ \partial_t \phi + \epsilon^{-2}\partial_x u &= -\epsilon^{-2}\omega\phi & \partial_x\phi &= -\epsilon^2\tau\partial_x\partial_t\phi - \tau\partial_x^2 u \end{aligned}$$

$$\Rightarrow \boxed{\epsilon^2\tau\partial_t^2 u + \partial_t u - \tau\partial_x^2 u = 0}$$

- **Goal:** Understanding singular limits: $\begin{cases} \text{Convergence} \\ \text{Arising of initial layers} \end{cases}$
- **Model problem:** D1P2 LB *equation* with $E\mathsf{f} = \frac{1}{2}(\mathsf{f}_1 + \mathsf{f}_2)$

$$\begin{aligned} \partial_t \mathsf{f}_1 - \epsilon^{-1}\partial_x \mathsf{f}_1 &= -\epsilon^{-2}\omega\left[\mathsf{f}_1 - E\mathsf{f}\right] = -\epsilon^{-2}\tfrac{\omega}{2}\left[\mathsf{f}_1 - \mathsf{f}_2\right] \\ \partial_t \mathsf{f}_2 + \epsilon^{-1}\partial_x \mathsf{f}_2 &= -\epsilon^{-2}\omega\left[\mathsf{f}_2 - E\mathsf{f}\right] = -\epsilon^{-2}\tfrac{\omega}{2}\left[\mathsf{f}_2 - \mathsf{f}_1\right] \end{aligned}$$

- **Reformulation:** 2×2 system $\longrightarrow$ *equivalent* scalar equation
  - Mass moment: $u = \mathsf{f}_1 + \mathsf{f}_2$, 1^{st} moment (flux): $\phi = \epsilon^{-1}(\mathsf{f}_2 - \mathsf{f}_1)$
  - Linear transformation $\mathsf{f}_1, \mathsf{f}_2 \leftrightarrow u, \phi$ leads to equivalent moment system:

$$\begin{aligned} \partial_t u + \partial_x \phi &= 0 & \partial_x\partial_t\phi &= -\partial_t^2 u \\ \partial_t \phi + \epsilon^{-2}\partial_x u &= -\epsilon^{-2}\omega\phi & \partial_x\phi &= -\epsilon^2\tau\partial_x\partial_t\phi - \tau\partial_x^2 u \end{aligned}$$

$$\Rightarrow \quad \boxed{\epsilon^2\tau\partial_t^2 u + \partial_t u - \tau\partial_x^2 u = 0}$$

- **BC:** bounce-back-type condition for $\mathsf{f} \longrightarrow$ hom. Dirichlet condition for u

$$\mathsf{f}_2(t, x_b) = -\mathsf{f}_1(t, x_b) \quad \Leftrightarrow \quad u(t, x_b) = 0$$

- **IC:** $\begin{bmatrix} \mathsf{f}_1(0,\cdot) \\ \mathsf{f}_2(0,\cdot) \end{bmatrix} \Leftrightarrow \begin{bmatrix} u(0,\cdot) \\ \phi(0,\cdot) \end{bmatrix} \Leftrightarrow \begin{bmatrix} u(0,\cdot) \\ \partial_t u(0,\cdot) = -\partial_x\phi(0,\cdot) \end{bmatrix}$

Reformulated LB equation	$\overset{\epsilon\downarrow 0}{\rightsquigarrow}$	**Target equation**
EQ: $\epsilon^2\tau\partial_t^2 u_\epsilon + \partial_t u_\epsilon - \tau\partial_x^2 u_\epsilon = 0$		EQ: $\partial_t u - \tau\partial_x^2 u = 0$
BC: $u_\epsilon(\cdot,0) = 0 \quad \wedge \quad u_\epsilon(\cdot,1) = 0$		BC: $u(\cdot,0) = 0 \ \wedge \ u(\cdot,1) = 0$
IC: $u_\epsilon(0,\cdot) = g \quad \wedge \quad \partial_t u_\epsilon(0,\cdot) = h$		IC: $u(0,\cdot) = g$

Compatible initialization: $h = \partial_t u(0,\cdot) = \tau\partial_x^2 u(0,\cdot) = \tau\partial_x^2 g(0,\cdot)$.

Reformulated LB equation	$\overset{\epsilon\downarrow 0}{\rightsquigarrow}$	**Target equation**
EQ: $\epsilon^2\tau\partial_t^2 u_\epsilon + \partial_t u_\epsilon - \tau\partial_x^2 u_\epsilon = 0$		EQ: $\partial_t u - \tau\partial_x^2 u = 0$
BC: $u_\epsilon(\cdot,0) = 0 \quad \wedge \quad u_\epsilon(\cdot,1) = 0$		BC: $u(\cdot,0) = 0 \ \wedge\ u(\cdot,1) = 0$
IC: $u_\epsilon(0,\cdot) = g \quad \wedge \quad \partial_t u_\epsilon(0,\cdot) = h$		IC: $u(0,\cdot) = g$

Compatible initialization: $h = \partial_t u(0,\cdot) = \tau\partial_x^2 u(0,\cdot) = \tau\partial_x^2 g(0,\cdot)$.

Fourier ansatz using $s_n(x) := \sin(n\pi x)$:

$$\begin{cases} \text{Initial cond.:} & \mathcal{L}^2(0,1) \ni g = \sum_n \alpha_n s_n, \qquad \mathcal{L}^2(0,1) \ni h = \sum_n \beta_n s_n \\ \text{Solutions:} & u_\epsilon(t,x) = \sum_n \sigma_{\epsilon,n}(t) s_n(x), \qquad u(t,x) = \sum_n \sigma_n(t) s_n(x) \end{cases}$$

Reformulated LB equation $\overset{\epsilon\downarrow 0}{\rightsquigarrow}$ **Target equation**

$$\left.\begin{aligned} &\text{EQ:} && \epsilon^2\tau\partial_t^2 u_\epsilon + \partial_t u_\epsilon - \tau\partial_x^2 u_\epsilon = 0 \\ &\text{BC:} && u_\epsilon(\cdot,0) = 0 \quad\wedge\quad u_\epsilon(\cdot,1) = 0 \\ &\text{IC:} && u_\epsilon(0,\cdot) = g \quad\wedge\quad \partial_t u_\epsilon(0,\cdot) = h \end{aligned}\right\} \quad \left\{\begin{aligned} &\text{EQ:} && \partial_t u - \tau\partial_x^2 u = 0 \\ &\text{BC:} && u(\cdot,0) = 0 \;\wedge\; u(\cdot,1) = 0 \\ &\text{IC:} && u(0,\cdot) = g \end{aligned}\right.$$

Compatible initialization: $h = \partial_t u(0,\cdot) = \tau\partial_x^2 u(0,\cdot) = \tau\partial_x^2 g(0,\cdot)$.

Fourier ansatz using $s_n(x) := \sin(n\pi x)$:

$$\left\{\begin{aligned} &\text{Initial cond.:} && \mathcal{L}^2(0,1) \ni g = \sum_n \alpha_n s_n, && \mathcal{L}^2(0,1) \ni h = \sum_n \beta_n s_n \\ &\text{Solutions:} && u_\epsilon(t,x) = \sum_n \sigma_{\epsilon,n}(t) s_n(x), && u(t,x) = \sum_n \sigma_n(t) s_n(x) \end{aligned}\right.$$

IVPs for the coefficient functions with $\lambda_n := \tau\pi^2 n^2$:

Perturbed problem $\overset{\epsilon\downarrow 0}{\rightsquigarrow}$ Limit problem

$$\left.\begin{aligned} &\text{EQ:} && \epsilon^2\tau\ddot{\sigma}_{\epsilon,n} + \dot{\sigma}_{\epsilon,n} + \lambda_n\sigma_{\epsilon,n} = 0 \\ &\text{IC:} && \sigma_{\epsilon,n}(0) = \alpha_n \;\wedge\; \dot{\sigma}_{\epsilon,n}(0) = \beta_n \end{aligned}\right\} \quad \left\{\begin{aligned} &\text{EQ:} && \dot{\sigma}_n + \lambda_n\sigma_n = 0 \\ &\text{IC:} && \sigma_n(0) = \alpha_n \end{aligned}\right.$$

- Estimate of Fourier coefficient functions: $|\sigma_{\epsilon,n}(t)| < 2|\alpha_n| + |\beta_n|\tau\epsilon^2$.
  Time derivative: $\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\epsilon,n}(t)\right| < |\alpha_n|\lambda_n + 2|\beta_n|$.
- Pointwise convergence of Fourier coefficients: $\sigma_{\epsilon,n}(t) \xrightarrow{\epsilon\downarrow 0} \sigma_n(t) = \alpha_n \mathrm{e}^{-\lambda_n t}$

- Estimate of Fourier coefficient functions: $|\sigma_{\epsilon,n}(t)| < 2|\alpha_n| + |\beta_n|\tau\epsilon^2$.
  Time derivative: $\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\epsilon,n}(t)\right| < |\alpha_n|\lambda_n + 2|\beta_n|$.
- Pointwise convergence of Fourier coefficients: $\sigma_{\epsilon,n}(t) \xrightarrow{\epsilon\downarrow 0} \sigma_n(t) = \alpha_n \mathrm{e}^{-\lambda_n t}$

- Set: $u_\epsilon(t,x) := \sum_n \sigma_{\epsilon,n}(t) s_n(x)$ (generally only solution in a weak sense)
- $\mathcal{L}^2$-convergence of Fourier-series: $g, h \in \mathcal{L}^2(0,1) \quad \Rightarrow \quad u_\epsilon(t,\cdot) \in \mathcal{L}^2(0,1)$
- Continuity in time: $u_\epsilon \in \mathcal{C}\big([0,\infty), \mathcal{L}^2(0,1)\big)$
- *Pointwise* convergence in *time* requiring only $\mathcal{L}^2$-regularity in *space*:

$$\|u_\epsilon(t,\cdot) - u(t,\cdot)\|_2 \xrightarrow{\epsilon\downarrow 0} 0$$

- Estimate of Fourier coefficient functions: $|\sigma_{\epsilon,n}(t)| < 2|\alpha_n| + |\beta_n|\tau\epsilon^2$.
  Time derivative: $\left|\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{\epsilon,n}(t)\right| < |\alpha_n|\lambda_n + 2|\beta_n|$.
- Pointwise convergence of Fourier coefficients: $\sigma_{\epsilon,n}(t) \xrightarrow{\epsilon\downarrow 0} \sigma_n(t) = \alpha_n \mathrm{e}^{-\lambda_n t}$

- Set: $u_\epsilon(t,x) := \sum_n \sigma_{\epsilon,n}(t) s_n(x)$ (generally only solution in a weak sense)
- $\mathcal{L}^2$-convergence of Fourier-series: $g, h \in \mathcal{L}^2(0,1) \quad \Rightarrow \quad u_\epsilon(t,\cdot) \in \mathcal{L}^2(0,1)$
- Continuity in time: $u_\epsilon \in \mathcal{C}\big([0,\infty), \mathcal{L}^2(0,1)\big)$
- *Pointwise* convergence in *time* requiring only $\mathcal{L}^2$-regularity in *space*:

$$\|u_\epsilon(t,\cdot) - u(t,\cdot)\|_2 \xrightarrow{\epsilon\downarrow 0} 0$$

- Convergence rate of Fourier coefficients: $\sup_{t\in[0,\infty)} \big|\sigma_{\epsilon,n}(t) - \sigma(t)\big| < C\epsilon^2$
  $+$ stronger regularity assumptions $\Rightarrow$ convergence rate for u_ϵ.
  (In particular: uniform convergence in time and space follows.)

- **Transfering** properties from $(\sigma_{\epsilon,n})_n$ to u_ϵ:
  Split Fourier series into $\begin{cases} \text{leading part} \leftarrow \text{finitely many terms, direct transfer.} \\ \text{tail} \leftarrow \text{infinitely many terms, but converging.} \end{cases}$

Convergence

Theorem: *If* $A := \sum_{n\geq 1} |\alpha_n|\lambda_n < \infty$ *and* $B := \sum_{n\geq 1} |\beta_n| < \infty$, *there exist constants* $C, \eta > 0$ *depending only on* τ *and on the initial data via* A *and* B *such that for all* $0 < \epsilon < \eta$*:*

$$\sup_{t\in[0,\infty)} \|u_\epsilon(t,\cdot) - u(t,\cdot)\|_\infty < C\epsilon^2.$$

Convergence

Theorem: *If* $A := \sum_{n\geq 1} |\alpha_n|\lambda_n < \infty$ *and* $B := \sum_{n\geq 1} |\beta_n| < \infty$, *there exist constants* $C, \eta > 0$ *depending only on* τ *and on the initial data via* A *and* B *such that for all* $0 < \epsilon < \eta$*:*

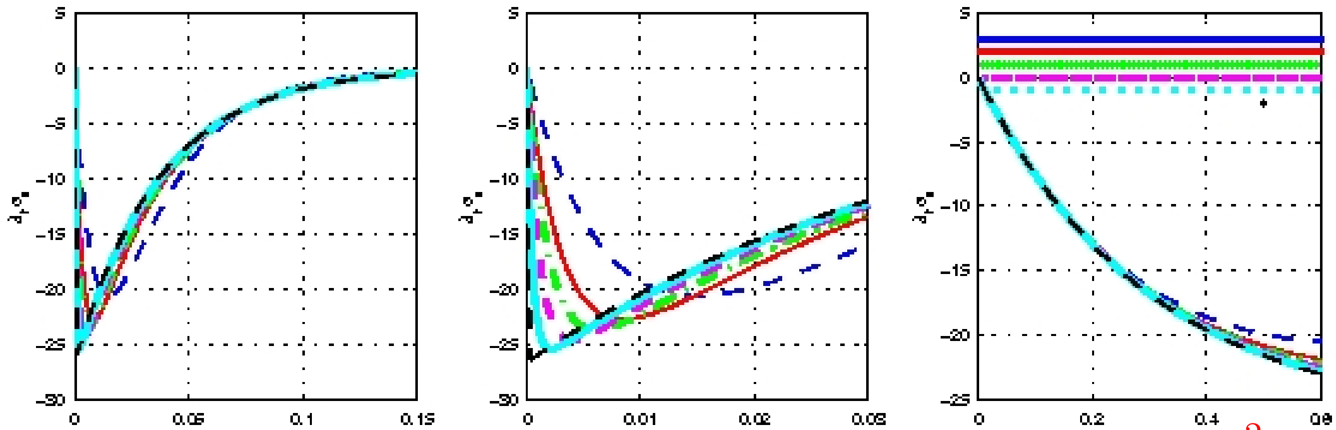
$$\sup_{t\in[0,\infty)} \|u_\epsilon(t,\cdot) - u(t,\cdot)\|_\infty < C\epsilon^2.$$

Remarks:

- $u_\epsilon(t,\cdot)$ not defined as solution of PDE but via Fourier series (convergence proof!).
- Assumptions on Fourier coefficients $(\alpha_n)_n, (\beta_n)_n \quad \Rightarrow$

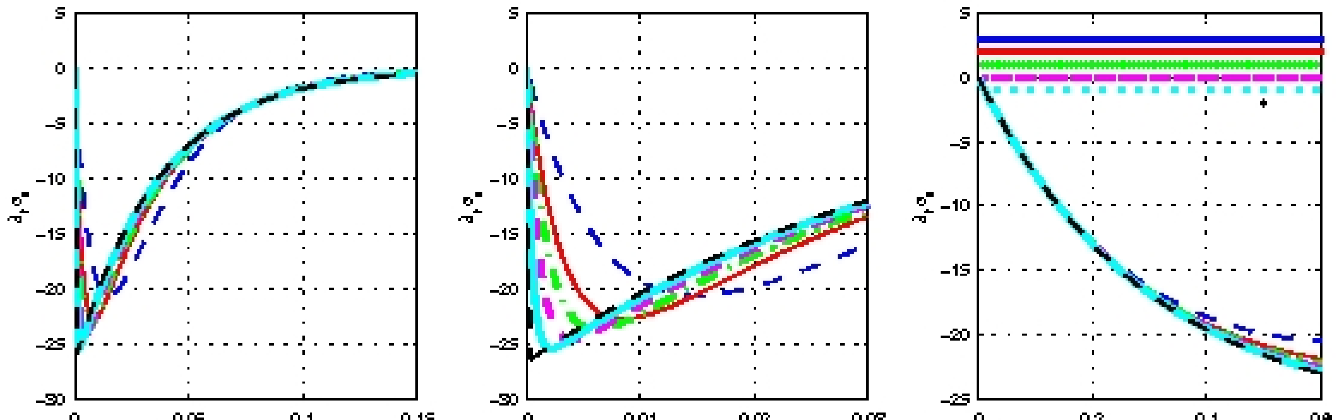
  regularity conditions: $g \in \mathcal{C}^2\big([0,1]\big), \quad h \in \mathcal{C}\big([0,1]\big)$.
- Convergence of $\partial_t u_\epsilon$: $\begin{cases} \text{generally: pointwise on } (0,\infty) \\ \text{compatible init.: uniformly on } [0,\infty) \\ \text{incompatible init.: uniformly on } [\theta,\infty) \quad \text{for arbitrary } \theta > 0 \end{cases}$
- Initial layer: $\begin{cases} \text{compensates incompatible initialization.} \\ \text{decays rapidly.} \end{cases}$

- **Ansatz:** $\sigma_\epsilon(t) = \sigma^{(0)}(\frac{t}{\epsilon^2}, t) \; + \; \epsilon^2 \sigma^{(2)}(\frac{t}{\epsilon^2}, t) \; + \; \dots$
- **Motivation:** consider plots of $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$ for different ϵ



slow time t | close-up | fast time t/ϵ^2

- **Ansatz:** $\sigma_\epsilon(t) = \sigma^{(0)}(\frac{t}{\epsilon^2}, t) \; + \; \epsilon^2 \sigma^{(2)}(\frac{t}{\epsilon^2}, t) \; + \; \dots$
- **Motivation:** consider plots of $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$ for different ϵ

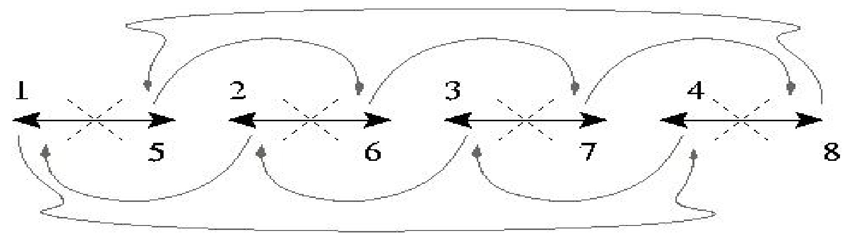


- **Outcome:** structure of order functions $\sigma^{(2k)}(t/\epsilon^2, t) = \underbrace{\mathrm{e}^{-\omega t/\epsilon^2}\phi^{(2k)}(t)}_{\text{irregular}} + \underbrace{\zeta^{(2k)}(t)}_{\text{regular}}$
- Hierarchic ODE-system defining the asymptotic order functions:

ϵ^0:	$\phi^{(0)} \equiv 0$	$\dot\zeta^{(0)} + \lambda\zeta^{(0)} = 0$ $\zeta^{(0)}(0) = \alpha$
ϵ^2:	$\dot\phi^{(2)} - \lambda\phi^{(2)} = 0$ $\phi^{(2)}(0) = \tau\dot\zeta^{(0)}(0) - \tau\beta$	$\dot\zeta^{(2)} + \lambda\zeta^{(2)} = -\tau\ddot\zeta^{(0)}$ $\zeta^{(2)}(0) = -\phi^{(2)}(0)$

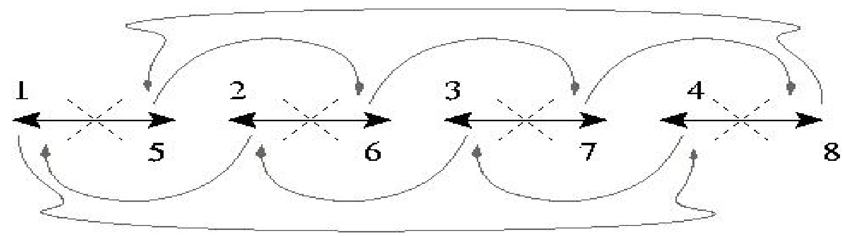
- **LB algorithm** $\rightarrow$ **explicit iteration:** $\mathsf{F}(n+1) = \mathbf{E}\,\mathsf{F}(n) = \mathbf{E}^{n+1}\,\mathsf{F}(0)$

- $$\mathbf{E} = \underbrace{\begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{R} \end{pmatrix}}_{\text{transport}} \underbrace{\begin{pmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \gamma\mathbf{I} & \delta\mathbf{I} \end{pmatrix}}_{\text{collision}}$$



- **LB algorithm** $\rightarrow$ **explicit iteration:** $\mathsf{F}(n+1) = \mathbf{E}\,\mathsf{F}(n) = \mathbf{E}^{n+1}\,\mathsf{F}(0)$

- $$\mathbf{E} = \underbrace{\begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{R} \end{pmatrix}}_{\text{transport}} \underbrace{\begin{pmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \gamma\mathbf{I} & \delta\mathbf{I} \end{pmatrix}}_{\text{collision}}$$



- Collision block: $\left[\begin{array}{ll} \alpha = 1 - \frac{1}{2}\omega(1+r) & \beta = \quad \frac{1}{2}\omega(1-r) \\ \gamma = \quad \frac{1}{2}\omega(1+r) & \delta = 1 - \frac{1}{2}\omega(1-r) \end{array} \right]$

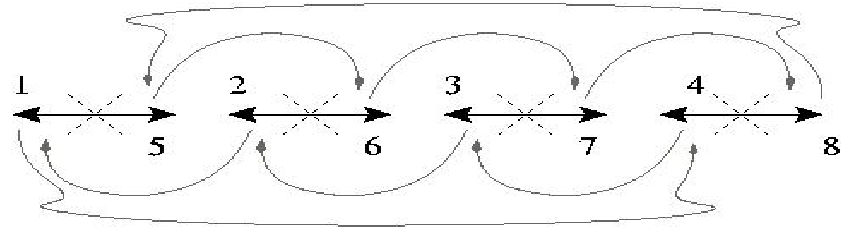
- Scaling:

  hyperbolic scaling
  $r = a,\ \Delta x = h,\ \Delta t = h$
  $\partial_t v + a\partial_x v = 0$

  parabolic scaling
  $r = ah,\ \Delta x = h,\ \Delta t = h^2$
  $\partial_t v + a\partial_x v - (\frac{1}{\omega} - \frac{1}{2})\partial_x^2 v = 0$

- **LB algorithm** $\longrightarrow$ **explicit iteration:** $\mathsf{F}(n+1) = \mathbf{E}\,\mathsf{F}(n) = \mathbf{E}^{n+1}\,\mathsf{F}(0)$

- $$\mathbf{E} = \underbrace{\begin{pmatrix} \mathbf{L} & 0 \\ 0 & \mathbf{R} \end{pmatrix}}_{\text{transport}} \underbrace{\begin{pmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \gamma\mathbf{I} & \delta\mathbf{I} \end{pmatrix}}_{\text{collision}}$$



- Collision block: $\begin{bmatrix} \alpha = 1 - \frac{1}{2}\omega(1+r) & \beta = \frac{1}{2}\omega(1-r) \\ \gamma = \frac{1}{2}\omega(1+r) & \delta = 1 - \frac{1}{2}\omega(1-r) \end{bmatrix}$

- Scaling:

  hyperbolic scaling
  $$r = a,\ \Delta x = h,\ \Delta t = h$$
  $$\partial_t v + a\partial_x v = 0$$

  parabolic scaling
  $$r = ah,\ \Delta x = h,\ \Delta t = h^2$$
  $$\partial_t v + a\partial_x v - (\tfrac{1}{\omega} - \tfrac{1}{2})\partial_x^2 v = 0$$

- **Computing eigenvalues of $\mathbf{E}$**

  $$\mathrm{spec}(\mathbf{L}) = \mathrm{spec}(\mathbf{R}) = \{w, w^2, \ldots, w^N\} \qquad w := \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$$

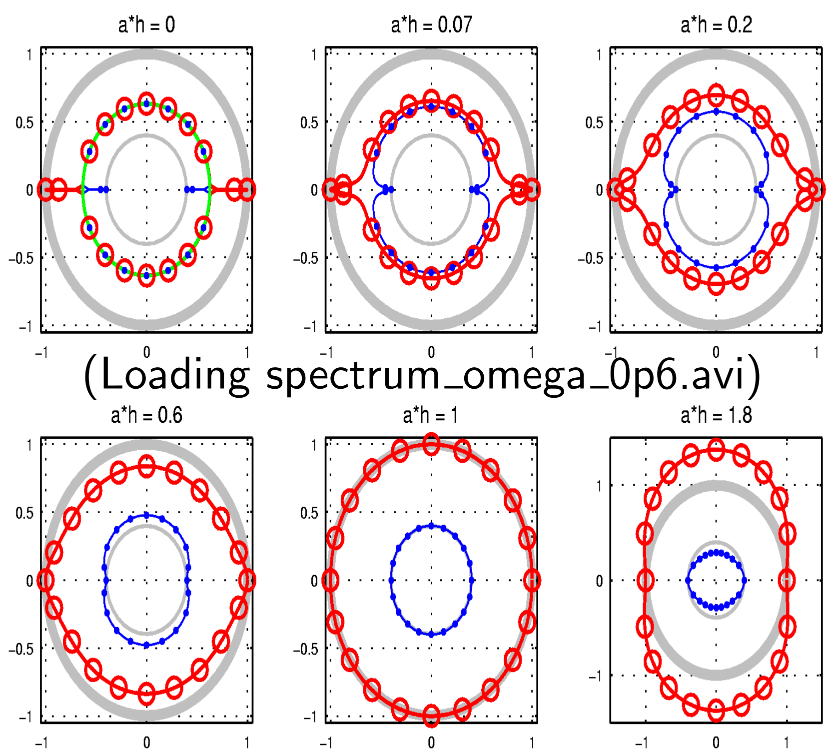
- Discrete Fourier transformation $\longrightarrow$ characteristic polynomial

  $$\lambda \mapsto \prod_{m=0}^{N-1} \underbrace{\left[(\alpha w^m - \lambda)(\delta \overline{w}^m - \lambda) - \beta\gamma\right]}_{\chi_{\omega,r}(\lambda;\frac{2\pi m}{N})}$$

$$\chi_{\omega,r}(\lambda;\phi) := \lambda^2 + \Big[(\omega-2)\cos(\phi) + \mathrm{i}\omega r\sin(\phi)\Big]\lambda + (1-\omega)$$

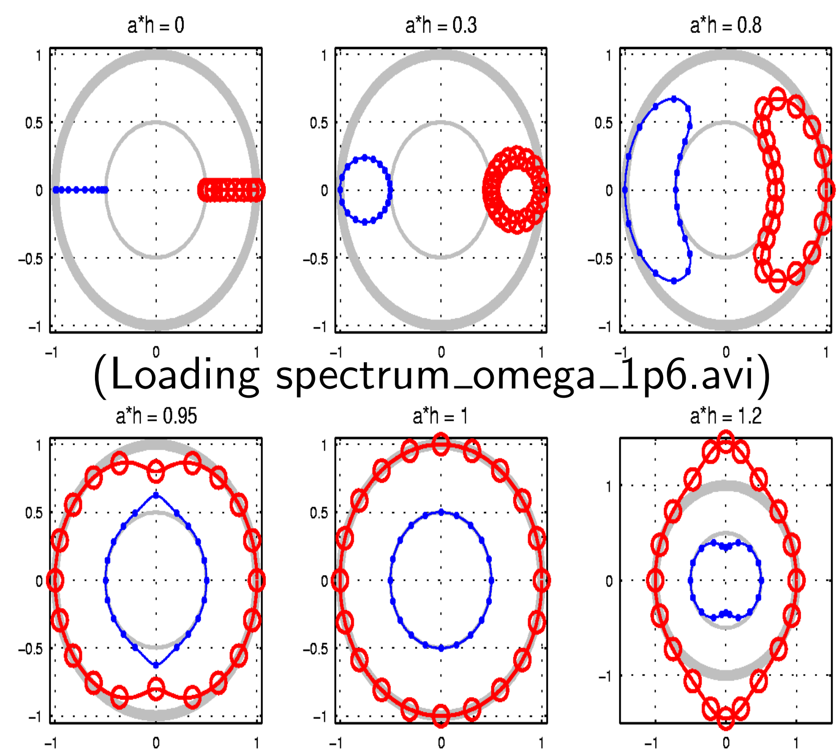
Spectral limit set:

$$\operatorname{spec}(\mathbf{E}) \subset \mathfrak{S}(\omega,r) := \Big\{\lambda\in\mathbb{C}\,\Big|\,\exists\phi\in[0,2\pi)\ \text{with}\ \chi_{\omega,r}(\lambda;\phi)=0\Big\}$$



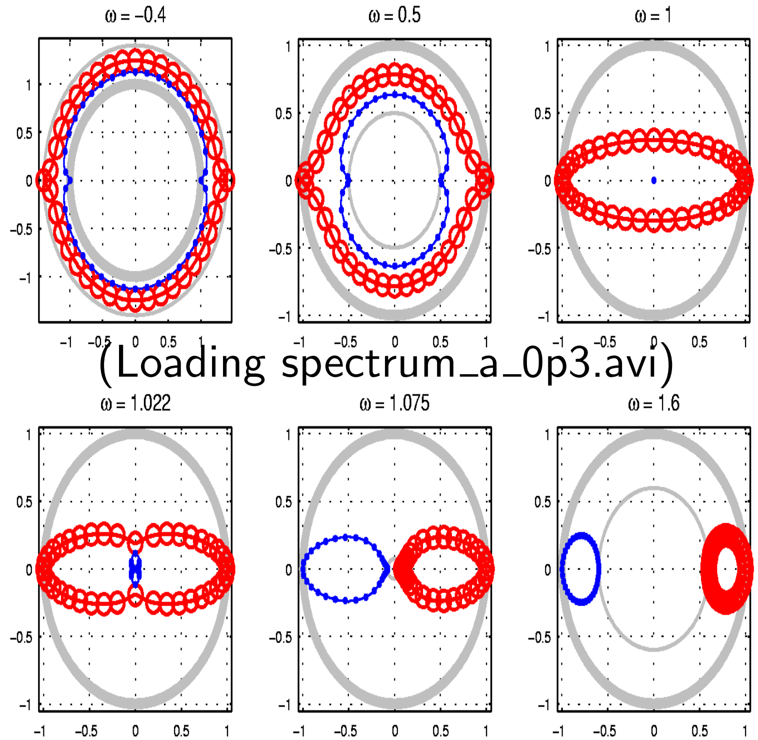
(Loading spectrum_omega_0p6.avi)

$\omega = 0.6$



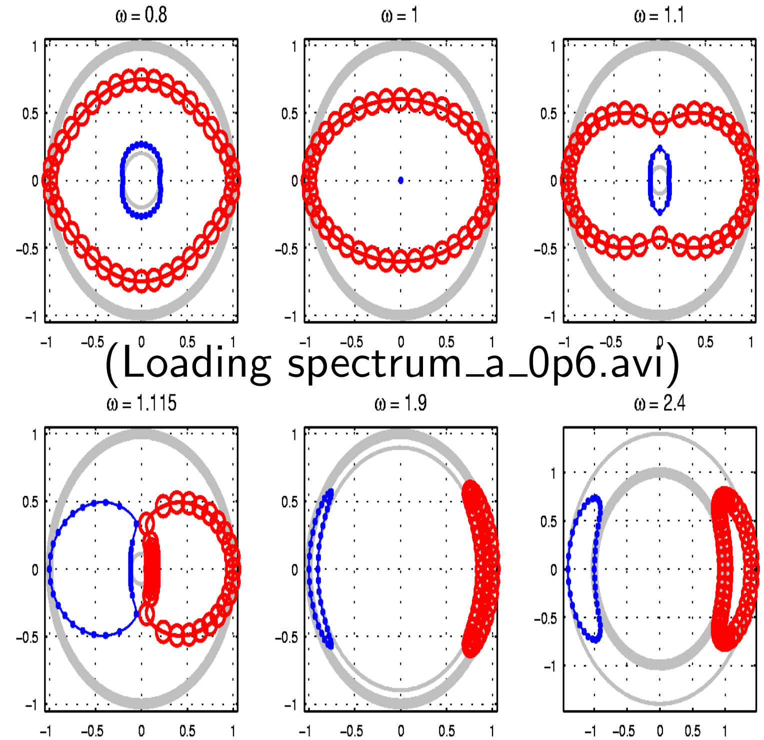
(Loading spectrum_omega_1p6.avi)

$\omega = 1.6$



(Loading spectrum_a_0p3.avi)

$r = 0.3$



(Loading spectrum_a_0p6.avi)

$r = 0.6$

Stability conditions:

- i) $\omega \in [0, 2]$ (general property)
- ii) $r \in [-1, 1]$ (specific property → CFL-condition)

Stability : $\exists K > 0 : \forall \text{grids}_h : \forall n \in \mathbb{N}_0, \quad : \|\mathbf{E}_h^n\|_h < K$

Stability over $[0, T]$: $\exists K > 0 : \forall\, \text{grids}_h : \forall\, n \in \mathbb{N}_0,\ n\Delta t_h \leq T : \|\mathbf{E}_h^n\|_h < K$

Stability over $[0,T]$: $\exists K > 0 : \ \forall\, \text{grids}_h \ : \ \forall\, n \in \mathbb{N}_0,\ n\Delta t_h \le T \ : \ \|\mathbf{E}_h^n\|_h < K$

CFL condition: **analytic** domain of depend. $\subset$ **numeric** domain of depend.

3-point stencil schemes: $\Rightarrow$ $\quad |a| \le \frac{\Delta x}{\Delta t} = \begin{cases} \frac{h}{h} = 1 & \text{hyperbolic scaling} \\ \frac{h}{h^2} = \frac{1}{h} & \text{parabolic scaling} \end{cases}$

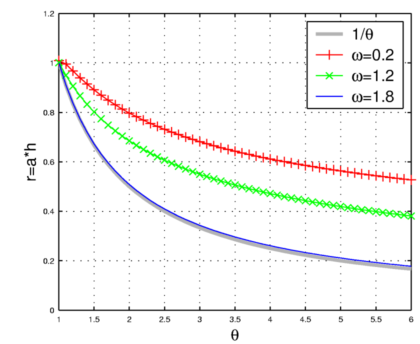
Stability over $[0,T]$: $\exists K > 0 : \forall\, \text{grids}_h : \forall\, n \in \mathbb{N}_0,\ n\Delta t_h \le T : \|\mathbf{E}_h^n\|_h < K$

CFL condition: **analytic** domain of depend. $\subset$ **numeric** domain of depend.

3-point stencil schemes: $\Rightarrow$ $$|a| \le \frac{\Delta x}{\Delta t} = \begin{cases} \frac{h}{h} = 1 & \text{hyperbolic scaling} \\ \frac{h}{h^2} = \frac{1}{h} & \text{parabolic scaling} \end{cases}$$

Standard FDM: **stability** $\Longleftrightarrow$ **CFL condition**

Result for LBM: **stability** $\overset{\text{D1P2, }\theta=1}{\Longleftrightarrow}$ / $\underset{\text{D1P3, }\theta>1}{\Longrightarrow}$ **CFL condition**



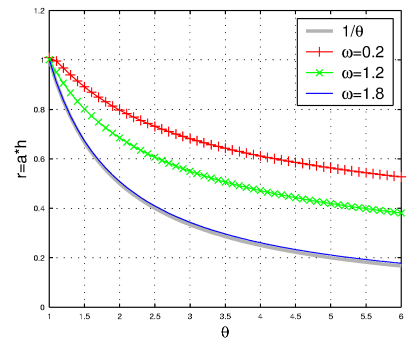
Stability over $[0, T]$: $\exists K > 0 : \forall$ grids$_h$ $: \forall n \in \mathbb{N}_0, \; n\Delta t_h \leq T : \|\mathbf{E}_h^n\|_h < K$

CFL condition: **analytic** domain of depend. $\subset$ **numeric** domain of depend.

3-point stencil schemes: $\Rightarrow$ $|a| \leq \frac{\Delta x}{\Delta t} = \begin{cases} \frac{h}{h} = 1 & \text{hyperbolic scaling} \\ \frac{h}{h^2} = \frac{1}{h} & \text{parabolic scaling} \end{cases}$

Standard FDM: **stability** $\Longleftrightarrow$ **CFL condition**

Result for LBM: **stability** $\overset{\text{D1P2, } \theta=1}{\Longleftrightarrow}$ / $\underset{\text{D1P3, } \theta>1}{\Longrightarrow}$ **CFL condition**



Theorem 1: $\mathfrak{S}(\omega, r) \subset \overline{D_1(0)} \quad \Longleftrightarrow \quad \begin{cases} \text{i)} & \omega \in [0, 2] \\ \text{ii)} & r \in [-1, 1] \end{cases}$ (for $\theta = 1$)

Theorem 2: *The advective-diffusive and the purely advective D1P2 lattice-Boltzmann scheme are stable w.r.t. the ℓ_2-norm if and only if $0 \leq \omega \leq 2$ and $-1 \leq r \leq 1$, or $\omega = 0$.*

Proof of theorem 1: Estimate zeros of $\chi_{\omega,r}(\lambda;\phi)$:

$$\lambda_{\omega,r}(\phi) = -\tfrac{1}{2}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega r\sin(\phi)\big] \pm \sqrt{\tfrac{1}{4}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega a\sin(\phi)\big]^2 - (1-\omega)}$$

Other idea $\longrightarrow$ consider special cases: $\omega = 1$ or $\phi \in \{0,\pi\}$ $\longrightarrow$ comparison function Theorem of *Rouché* $\longrightarrow$ *general case*. ■

Proof of theorem 1: Estimate zeros of $\chi_{\omega,r}(\lambda;\phi)$:

$$\lambda_{\omega,r}(\phi) = -\tfrac{1}{2}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega r\sin(\phi)\big] \pm \sqrt{\tfrac{1}{4}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega a\sin(\phi)\big]^2 - (1-\omega)}$$

Other idea $\longrightarrow$ consider special cases: $\omega = 1$ or $\phi \in \{0,\pi\}$ $\longrightarrow$ comparison function. Theorem of *Rouché* $\longrightarrow$ *general case*. ■

Proof of theorem 2:

- Discrete Fourier trafo & permutation of indices:

$$\mathbf{E} = \operatorname{blockdiag}\big(M(\phi)\big)_{\phi\in\frac{2\pi}{N}\{0,1,\ldots,N-1\}} \quad \text{with} \quad M(\phi) = \begin{pmatrix} \alpha e^{\mathrm{i}\phi} & \beta e^{\mathrm{i}\phi} \\ \gamma e^{-\mathrm{i}\phi} & \delta e^{-\mathrm{i}\phi} \end{pmatrix}$$

$$\Rightarrow \qquad \|\mathbf{E}^n\|_2 \le \sup_{\phi\in[0,2\pi]} \|M^n(\phi)\|_2$$

Proof of theorem 1: Estimate zeros of $\chi_{\omega,r}(\lambda;\phi)$:

$$\lambda_{\omega,r}(\phi) = -\tfrac{1}{2}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega r\sin(\phi)\big] \pm \sqrt{\tfrac{1}{4}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega a\sin(\phi)\big]^2 - (1-\omega)}$$

Other idea $\longrightarrow$ consider special cases: $\omega = 1$ or $\phi \in \{0, \pi\}$ $\longrightarrow$ comparison function.
Theorem of *Rouché* $\longrightarrow$ *general case*. ■

Proof of theorem 2:

- Discrete Fourier trafo & permutation of indices:

$$\mathbf{E} = \mathrm{blockdiag}\big(M(\phi)\big)_{\phi \in \frac{2\pi}{N}\{0,1,\ldots,N-1\}} \quad \text{with} \quad M(\phi) = \begin{pmatrix} \alpha \mathrm{e}^{\mathrm{i}\phi} & \beta \mathrm{e}^{\mathrm{i}\phi} \\ \gamma \mathrm{e}^{-\mathrm{i}\phi} & \delta \mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix}$$

$$\Rightarrow \qquad \|\mathbf{E}^n\|_2 \le \sup_{\phi \in [0,2\pi]} \|M^n(\phi)\|_2$$

- Family of continuous functions: $n \in \mathbb{N}$: $\quad f_n : [0, 2\pi] \to \mathbb{R}, \quad f_n(\phi) := \|M^n(\phi)\|_2$

Proof of theorem 1: Estimate zeros of $\chi_{\omega,r}(\lambda;\phi)$:

$$\lambda_{\omega,r}(\phi) = -\tfrac{1}{2}\big[(\omega-2)\cos(\phi) + \mathrm{i}\omega r\sin(\phi)\big] \pm \sqrt{\tfrac{1}{4}\big[(\omega-2)\cos(\phi)+\mathrm{i}\omega a\sin(\phi)\big]^2 - (1-\omega)}$$

Other idea $\longrightarrow$ consider special cases: $\omega = 1$ or $\phi \in \{0,\pi\}$ $\longrightarrow$ comparison function.
Theorem of *Rouché* $\longrightarrow$ *general case*. ■

Proof of theorem 2:

- Discrete Fourier trafo & permutation of indices:

$$\mathbf{E} = \mathrm{blockdiag}\big(M(\phi)\big)_{\phi\in\frac{2\pi}{N}\{0,1,\ldots,N-1\}} \quad \text{with} \quad M(\phi) = \begin{pmatrix} \alpha e^{\mathrm{i}\phi} & \beta e^{\mathrm{i}\phi} \\ \gamma e^{-\mathrm{i}\phi} & \delta e^{-\mathrm{i}\phi} \end{pmatrix}$$

$$\Rightarrow \qquad \|\mathbf{E}^n\|_2 \le \sup_{\phi\in[0,2\pi]} \|M^n(\phi)\|_2$$

- Family of continuous functions: $n\in\mathbb{N}$: $\quad f_n : [0,2\pi] \longrightarrow \mathbb{R}, \quad f_n(\phi) := \|M^n(\phi)\|_2$
- Theorem 1 $\rho\big(M(\phi)\big) \le 1$ & diagonalizibility of $M(\phi)$:
  $\Rightarrow$ pointwise boundedness of $(f_n)_{n\in\mathbb{N}}$, i.e.:

$$\exists\, C_\phi > 0, \quad \forall\, n\in\mathbb{N}: \quad \sup_{n\in\mathbb{N}} \|M^n(\phi)\|_2 = \sup_{n\in\mathbb{N}} \|f_n(\phi)\|_2 < C_\phi$$

Proof of theorem 1: Estimate zeros of $\chi_{\omega,r}(\lambda;\phi)$:

$$\lambda_{\omega,r}(\phi) = -\tfrac{1}{2}\big[(\omega-2)\cos(\phi)+\mathrm{i}\omega r\sin(\phi)\big] \pm \sqrt{\tfrac{1}{4}\big[(\omega-2)\cos(\phi)+\mathrm{i}\omega a\sin(\phi)\big]^2-(1-\omega)}$$

Other idea $\longrightarrow$ consider special cases: $\omega = 1$ or $\phi \in \{0,\pi\}$ $\longrightarrow$ comparison function. Theorem of *Rouché* $\longrightarrow$ *general case*. ■

Proof of theorem 2:

- Discrete Fourier trafo & permutation of indices:

$$\mathbf{E} = \mathrm{blockdiag}\big(M(\phi)\big)_{\phi\in\frac{2\pi}{N}\{0,1,\dots,N-1\}} \quad \text{with} \quad M(\phi) = \begin{pmatrix} \alpha e^{\mathrm{i}\phi} & \beta e^{\mathrm{i}\phi} \\ \gamma e^{-\mathrm{i}\phi} & \delta e^{-\mathrm{i}\phi} \end{pmatrix}$$

$$\Rightarrow \qquad \|\mathbf{E}^n\|_2 \le \sup_{\phi\in[0,2\pi]} \|M^n(\phi)\|_2$$

- Family of continuous functions: $n\in\mathbb{N}$: $\quad f_n : [0,2\pi]\longrightarrow\mathbb{R}, \quad f_n(\phi) := \|M^n(\phi)\|_2$
- Theorem 1 $\rho\big(M(\phi)\big)\le 1$ & diagonalizibility of $M(\phi)$:
  $\Rightarrow$ pointwise boundedness of $(f_n)_{n\in\mathbb{N}}$, i.e.:

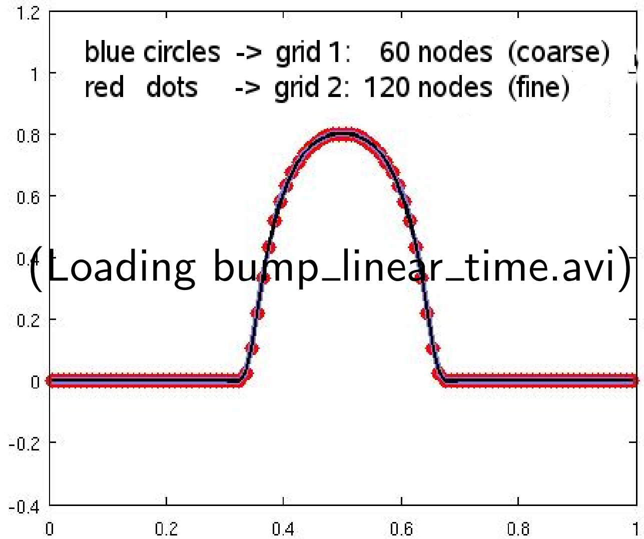
$$\exists\, C_\phi > 0, \quad \forall\, n\in\mathbb{N}: \quad \sup_{n\in\mathbb{N}} \|M^n(\phi)\|_2 = \sup_{n\in\mathbb{N}} \|f_n(\phi)\|_2 < C_\phi$$

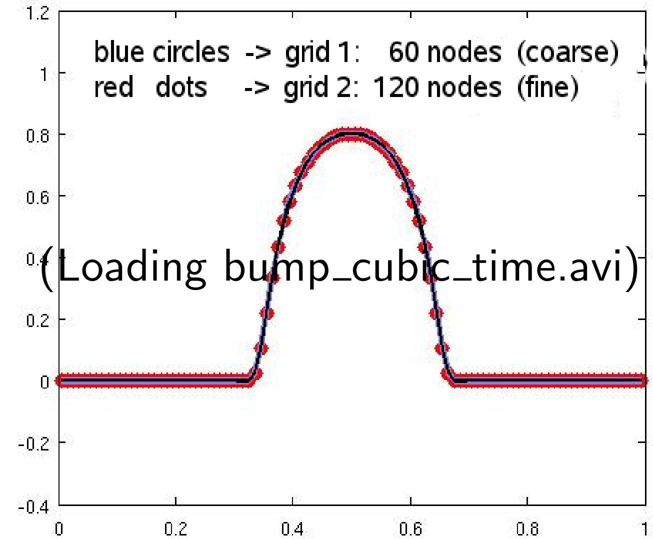
- *Principle of uniform boundedness:* $\Rightarrow$ local boundedness.
- Compactness of $[0,2\pi]$ $\Rightarrow$ global boundedness. ■

Long time behavior of the advective D1P2 LB scheme

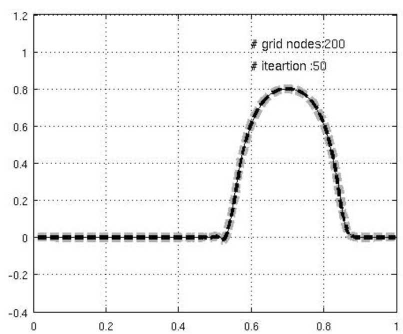
Observation: linear time-scale (advection) $\leftrightarrow$ cubic time-scale (dispersion)

$$t_{\text{lin}} = n_1 h_1 = n_2 h_2 \qquad\qquad t_{\text{cub}} = n_1 h_1^3 = n_2 h_2^3$$

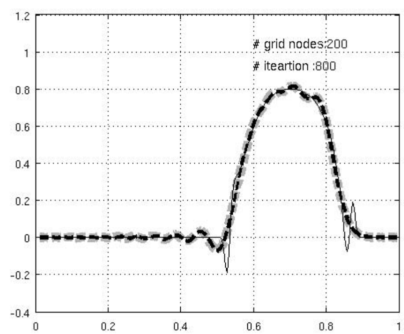




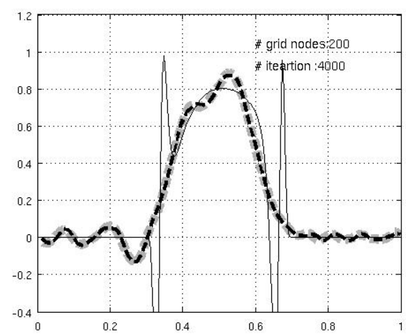
Prediction: comparison regular $\leftrightarrow$ twoscale expansion (200 nodes)



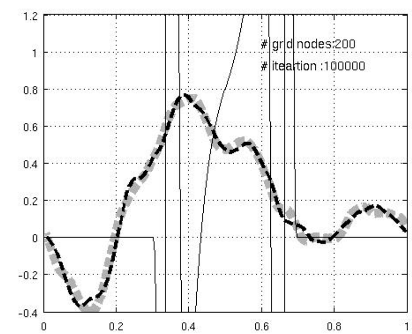
50 iterations



800 iterations



4000 iterations



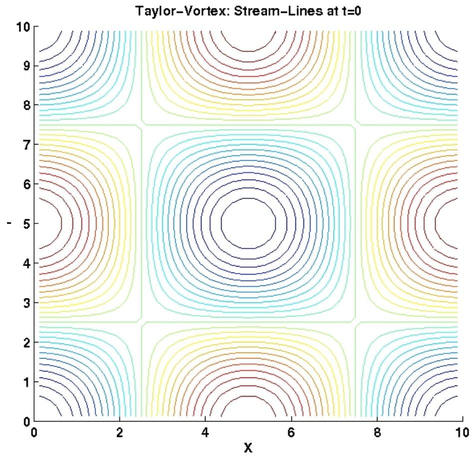
100000 iterations

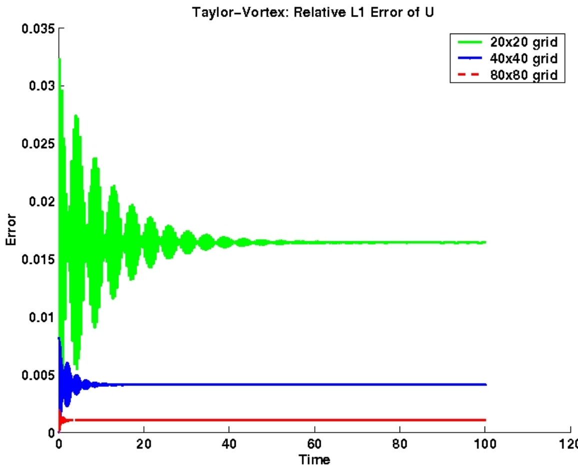
Σummary

- LBM: PDE solver inspired by pseudo-particle dynamics (collision/transport step)
- Motivation: lack of solid understanding despite of rich engineering experience
  $\longrightarrow$ elimination of numerical artefacts, basis for systematic extensions
- Important analytic tool: asymptotic expansions
- Presented results:
  - better comprehension of initial layers (generation, long time impact)
  - exemplarily: stability properties of an LB model algorithm

Supplements

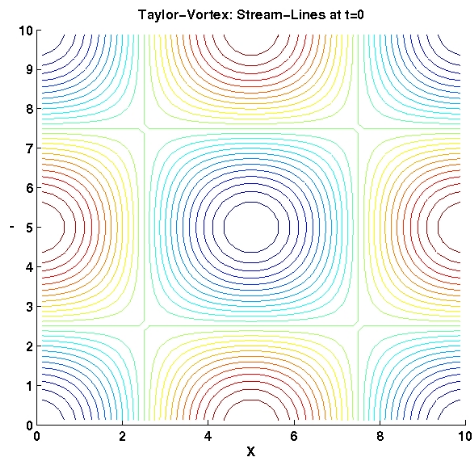
Example: Observation of an initial layer (simulating decaying eigenmode of Stokes operator)

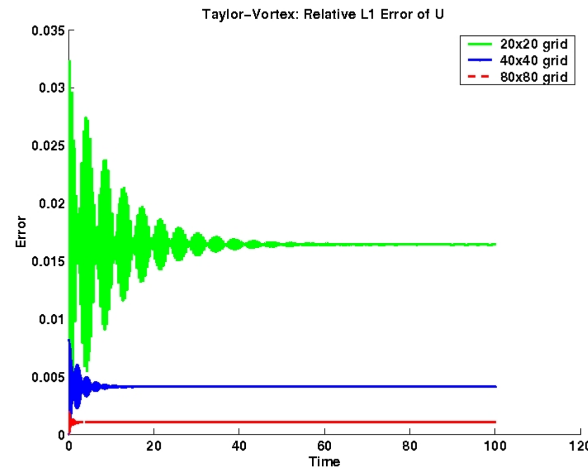




- Unusual behavior: rapid *decrease* of error instead of *growth*.
- Composition of numeric error displaying several time scales:

Example: Observation of an initial layer (simulating decaying eigenmode of Stokes operator)

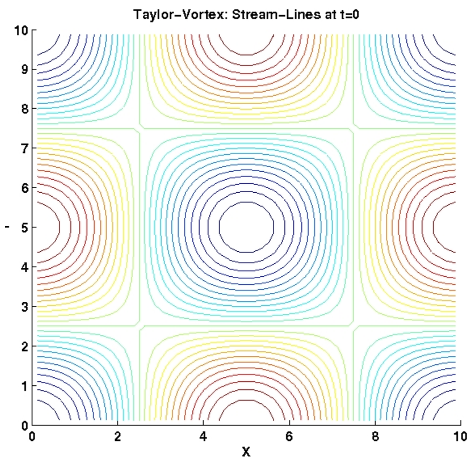


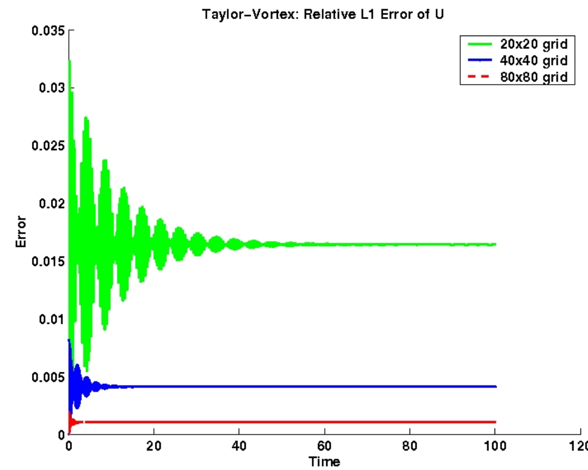


- Unusual behavior: rapid *decrease* of error instead of *growth*.
- Composition of numeric error displaying several time scales:

feature	$t(n)$	time scale	interpretation	evolution governed by ...
plateau	nh^2	slow time (plotted)	standard discretization error	inhomogeneous Stokes eq.
'beat-bellies'	nh	fast time	initial layer of FVBE	'wave-like' PDE (pseudo-sound)
decay	n	discrete time	discrete initial layer	$\|1-\omega\|^n$
oscillations	n	discrete time	discrete initial layer	$(-1)^n$

Example: Observation of an initial layer (simulating decaying eigenmode of Stokes operator)



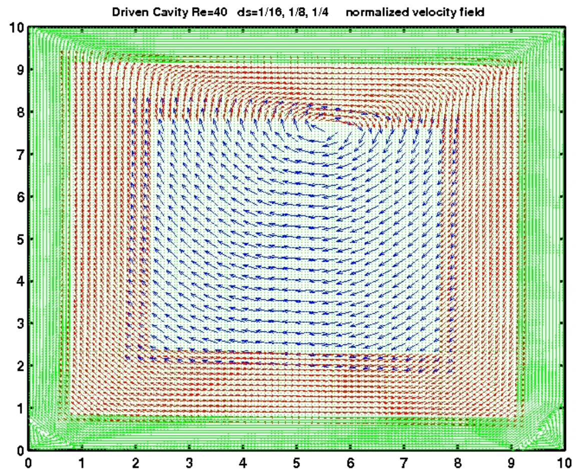


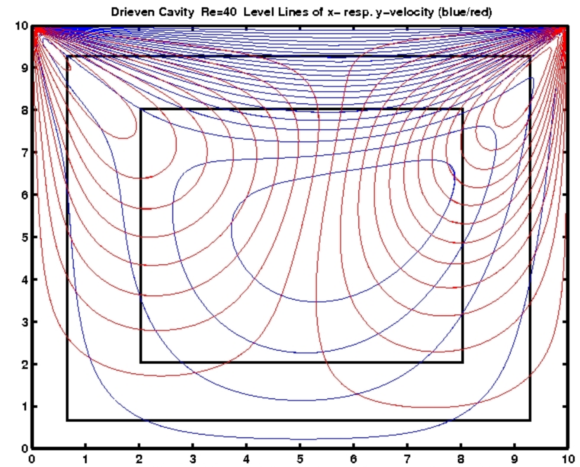
- Unusual behavior: rapid *decrease* of error instead of *growth*.
- Composition of numeric error displaying several time scales:

feature	$t(n)$	time scale	interpretation	evolution governed by ...
plateau	nh^2	slow time (plotted)	standard discretization error	inhomogeneous Stokes eq.
'beat-bellies'	nh	fast time	initial layer of FVBE	'wave-like' PDE (pseudo-sound)
decay	n	discrete time	discrete initial layer	$\|1-\omega\|^n$
oscillations	n	discrete time	discrete initial layer	$(-1)^n$

Incipient questions: consistency (traditional approach via Chapman-Enskog expansion)
convergence (requires stability)
further properties (multiple time scales, scaling, numerical layers)

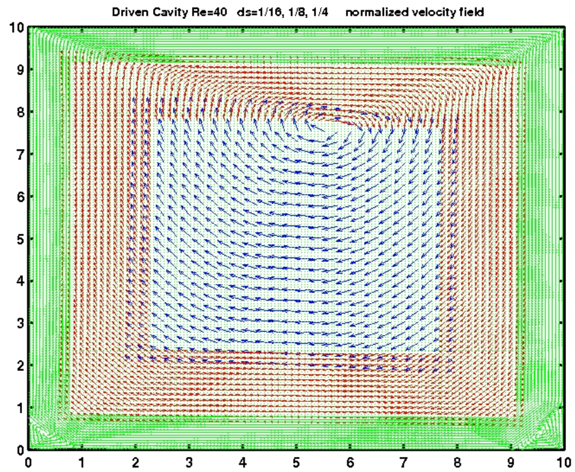
Embedding into context of further problems: e.g. grid coupling

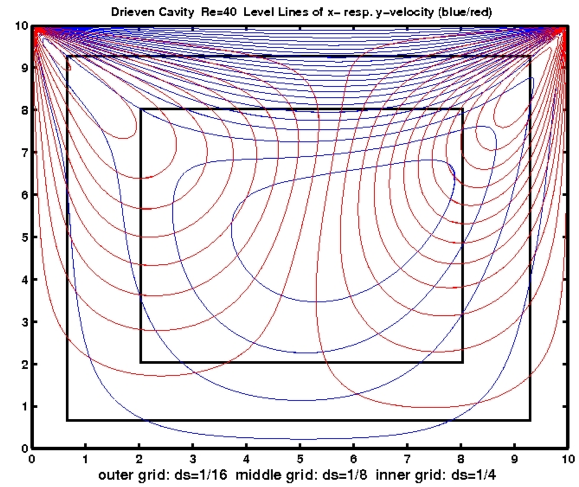




Domain decomposition ⟶ coupling conditions for target equation ⟶ translation into interface conditions for LB primary variables ⟶ interface layers in the case of incompatibilities

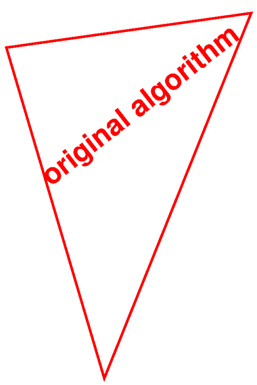
Embedding into context of further problems: e.g. grid coupling



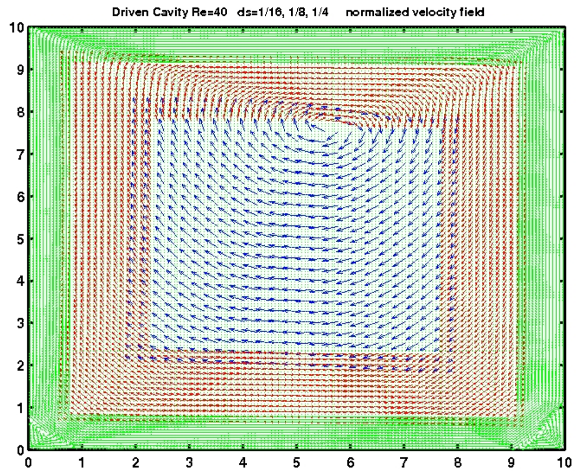


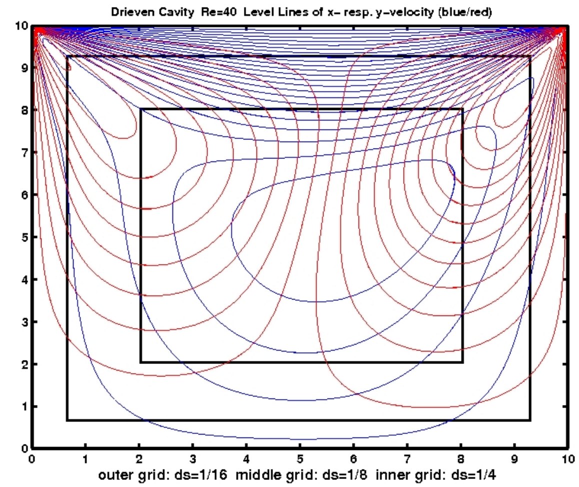
Domain decomposition ⟶ coupling conditions for target equation ⟶ translation into interface conditions for LB primary variables ⟶ interface layers in the case of incompatibilities

General strategy:



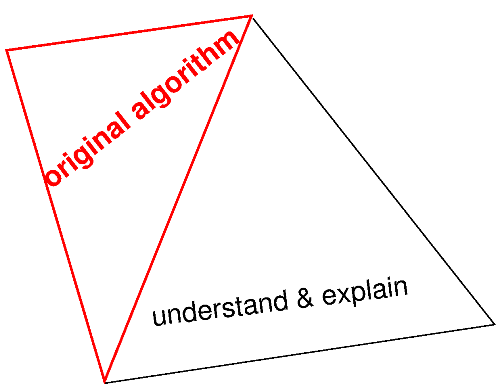
Embedding into context of further problems: e.g. grid coupling



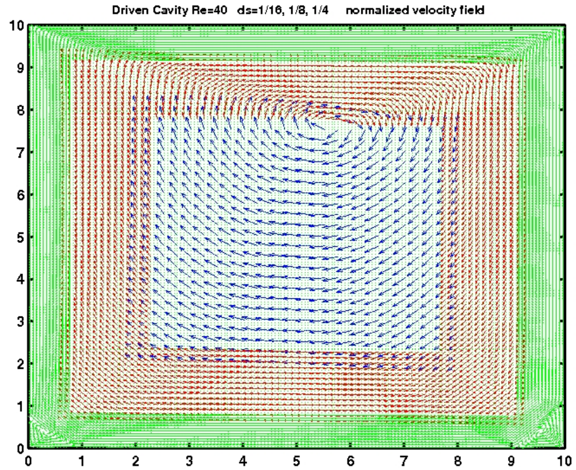


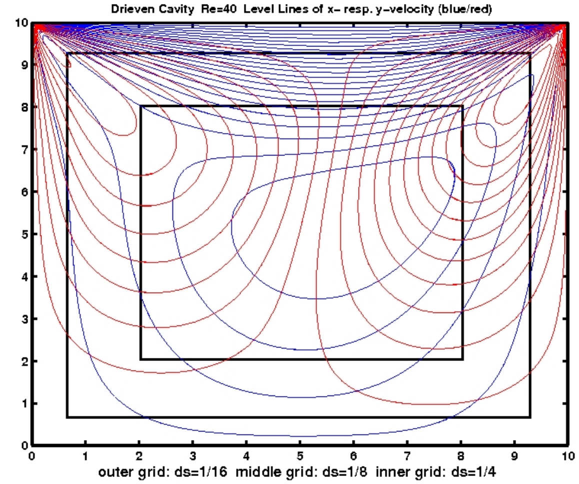
Domain decomposition → coupling conditions for target equation → translation into interface conditions for LB primary variables → interface layers in the case of incompatibilities

General strategy:



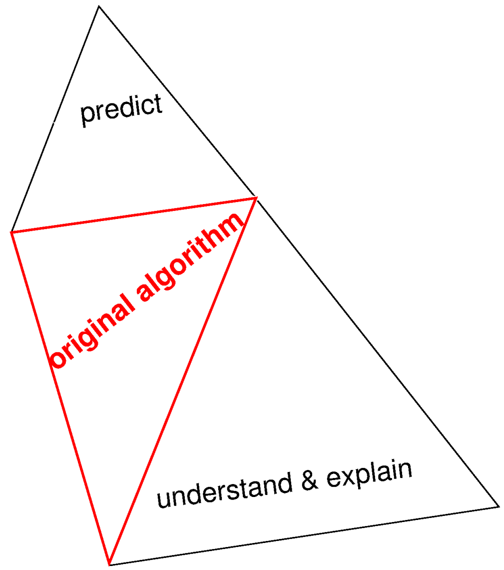
Embedding into context of further problems: e.g. grid coupling



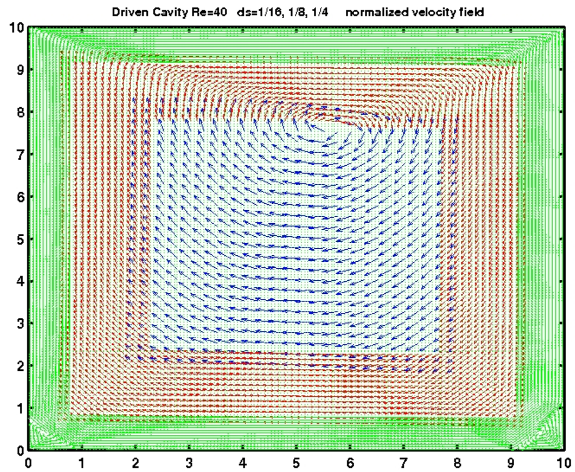


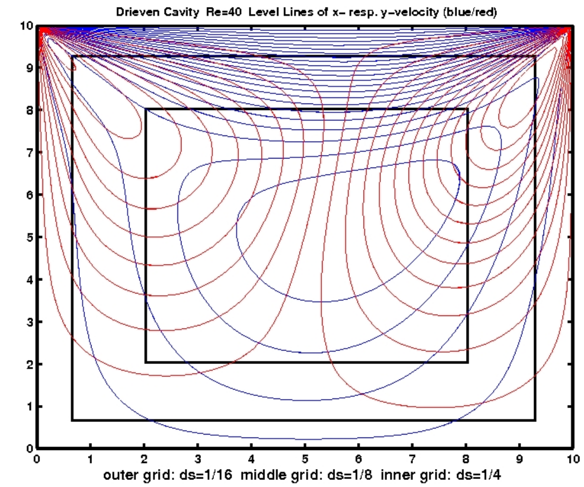
Domain decomposition → coupling conditions for target equation → translation into interface conditions for LB primary variables → interface layers in the case of incompatibilities

General strategy:



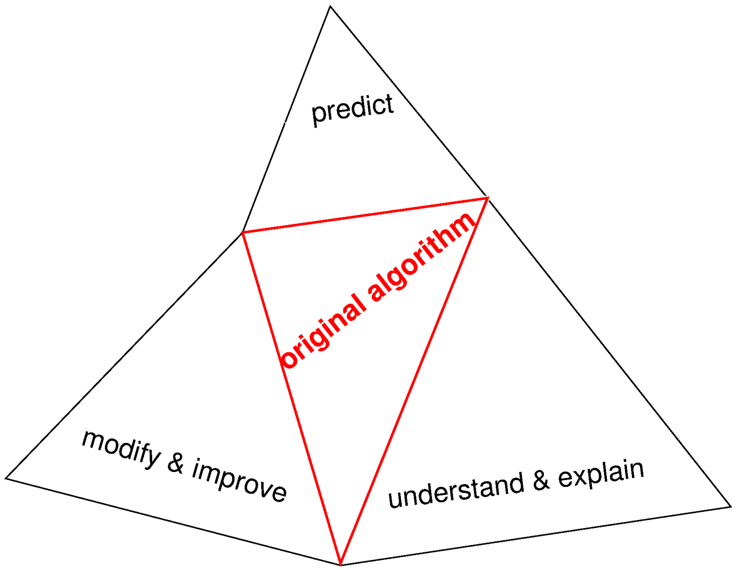
Embedding into context of further problems: e.g. grid coupling



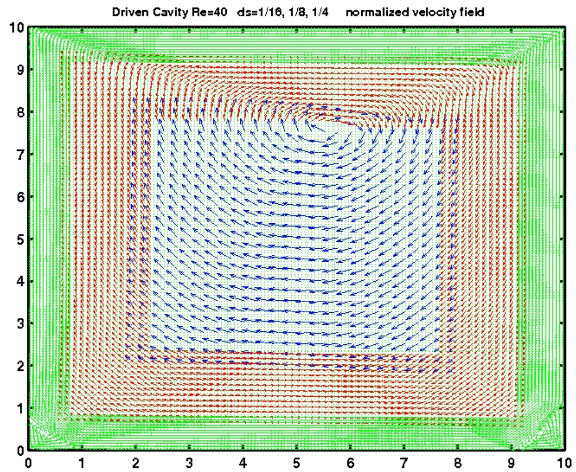


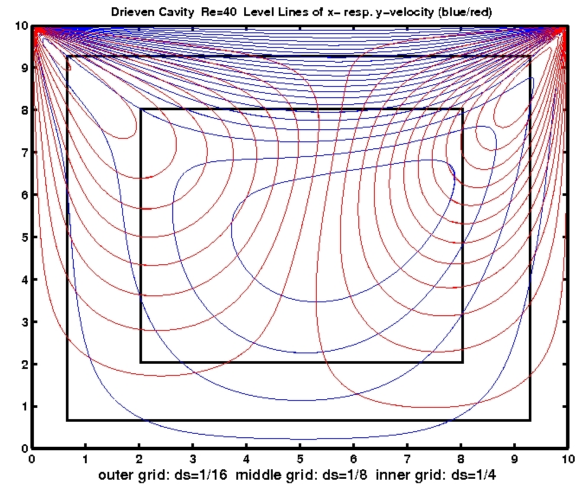
Domain decomposition → coupling conditions for target equation → translation into interface conditions for LB primary variables → interface layers in the case of incompatibilities

General strategy:



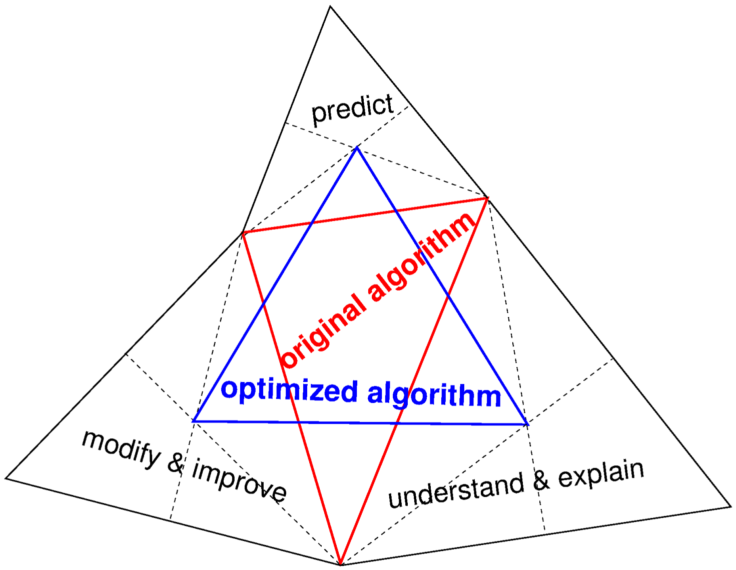
Embedding into context of further problems: e.g. grid coupling





Domain decomposition $\longrightarrow$ coupling conditions for target equation $\longrightarrow$ translation into interface conditions for LB primary variables $\longrightarrow$ interface layers in the case of incompatibilities

General strategy:



- **General setup:** $\eta \in \mathcal{H} \subset (0,1]: \quad A_\eta : X_\eta \to X_\eta, \qquad A_\eta x_\eta = 0$
  Wanted: asymptotic behavior of x_η for $\eta \to 0$

– Singular limit: $x_\eta \xrightarrow{\eta\to 0} \overline{x}_0 \in X_0$

while $A_\eta \xrightarrow{\text{formally}} A_0 : X_0 \to X_0$ but $A_0 x_0 = 0$ ill-posed

- **General setup:** $\eta \in \mathcal{H} \subset (0,1]: \quad A_\eta : X_\eta \to X_\eta, \qquad A_\eta x_\eta = 0$
  Wanted: asymptotic behavior of x_η for $\eta \to 0$

– Singular limit: $x_\eta \xrightarrow{\eta \to 0} \overline{x}_0 \in X_0$

while $A_\eta \xrightarrow{\text{formally}} A_0 : X_0 \to X_0$ but $A_0 x_0 = 0$ ill-posed

- **Comparison function** $\to$ ansatz: *regular expansion*

$$y_\eta^{[n]} := y^{(0)} + \eta y^{(2)} + \ldots + \eta^n y^{(n)} \qquad \text{with} \quad y^{(k)} \in X_0$$

– Alternatively: *irregular expansion:* $y^{(k)} = y_\eta^{(k)} \in \begin{cases} X_\eta & \text{(discrete coefficient functions)} \\ X_0 & \text{(e.g. multiscale expansion)} \end{cases}$

- **General setup:** $\eta \in \mathcal{H} \subset (0,1]: \quad A_\eta : X_\eta \to X_\eta, \qquad A_\eta x_\eta = 0$
  Wanted: asymptotic behavior of x_η for $\eta \to 0$

– Singular limit: $x_\eta \xrightarrow{\eta\to 0} \overline{x}_0 \in X_0$

while $A_\eta \xrightarrow{\text{formally}} A_0 : X_0 \to X_0$ but $A_0 x_0 = 0$ ill-posed

- **Comparison function** $\to$ ansatz: *regular expansion*

$$y_\eta^{[n]} := y^{(0)} + \eta y^{(2)} + \ldots + \eta^n y^{(n)} \qquad \text{with} \quad y^{(k)} \in X_0$$

– Alternatively: *irregular expansion:* $y^{(k)} = y_\eta^{(k)} \in \begin{cases} X_\eta & \text{(discrete coefficient functions)} \\ X_0 & \text{(e.g. multiscale expansion)} \end{cases}$

– Minimize **residue:** $r_\eta^{[n]} := A_\eta\big(R_\eta y_\eta^{[n]}\big) \qquad R_\eta : X_0 \to X_\eta$ (restriction/projection)

e.g. $r_\eta^{[n]} = \mathrm{O}(\eta^n) \quad \Rightarrow \quad n$ "consistency order"

- **General setup:** $\eta \in \mathcal{H} \subset (0,1]: \quad A_\eta : X_\eta \to X_\eta, \qquad A_\eta x_\eta = 0$
  Wanted: asymptotic behavior of x_η for $\eta \to 0$

– Singular limit: $x_\eta \xrightarrow{\eta \to 0} \overline{x}_0 \in X_0$

while $A_\eta \xrightarrow{\text{formally}} A_0 : X_0 \to X_0$ but $A_0 x_0 = 0$ ill-posed

- **Comparison function** $\to$ ansatz: *regular expansion*

$$y_\eta^{[n]} := y^{(0)} + \eta y^{(2)} + \ldots + \eta^n y^{(n)} \qquad \text{with } y^{(k)} \in X_0$$

– Alternatively: *irregular expansion:* $y^{(k)} = y_\eta^{(k)} \in \begin{cases} X_\eta & \text{(discrete coefficient functions)} \\ X_0 & \text{(e.g. multiscale expansion)} \end{cases}$

– Minimize **residue:** $r_\eta^{[n]} := A_\eta\big(R_\eta y_\eta^{[n]}\big) \qquad R_\eta : X_0 \to X_\eta$ (restriction/projection)

e.g. $r_\eta^{[n]} = \mathrm{O}(\eta^n) \quad \Rightarrow \quad n$ "consistency order"

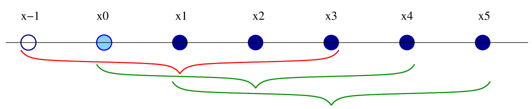
- **Asymptotic similarity:**

$$\begin{aligned} \big\| R_\eta y_\eta^{[n]} - x_\eta \big\|_{X_\eta} &= \big\| (A_\eta^{-1} \circ A_\eta) R_\eta y_\eta^{[n]} - (A_\eta^{-1} \circ A_\eta) x_\eta \big\|_{X_\eta} \\ &= \big\| A_\eta^{-1} r_\eta^{[n]} - A_\eta^{-1} 0 \big\|_{X_\eta} \leq \mathrm{Lip}_{A_\eta^{-1}} \big\| r_\eta^{[n]} \big\|_{X_\eta} \xrightarrow[\eta \to 0]{\text{stability}} 0 \end{aligned}$$

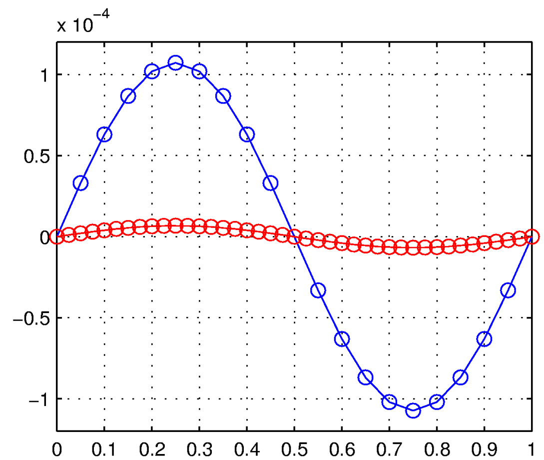
- **Remarks:** non-uniqueness of order functions (high order regular $y^{(n)}$, irregular $y_\eta^{(k)}$), ambiguity of consistency order $\eta^\alpha A_\eta x_\eta = 0 \Leftrightarrow A_\eta x_\eta = 0$, crude standard estimate

Analysis of a numerical boundary layer

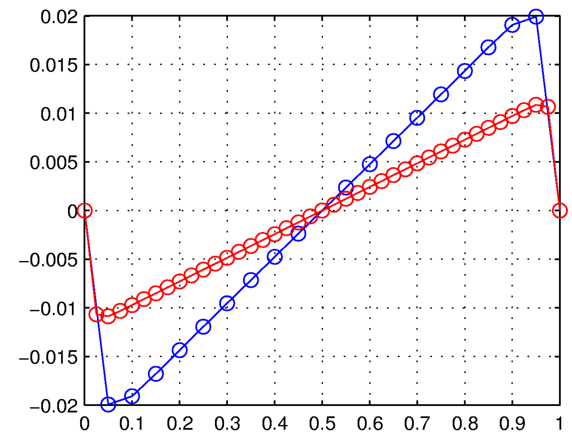
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = -4\pi^2 \sin(2\pi x) \quad +\text{BCs} \qquad \longrightarrow u(x) = \sin(2\pi x)$$



- 5-point stencil discretizing $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- incompatibility at the boundary
- nearest neighbor extrapolation in ghost node



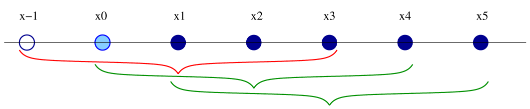
Error (periodic BCs)



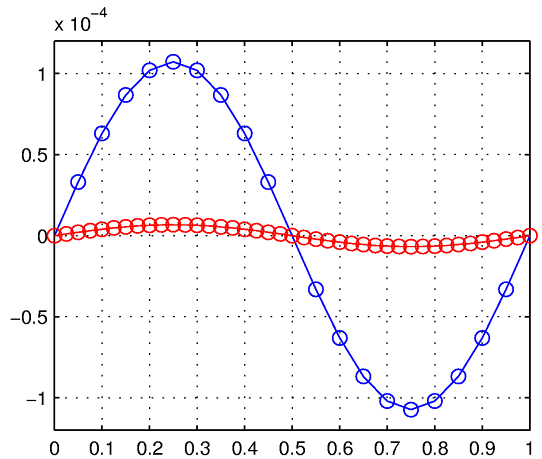
Error (homog. Dirichlet BCs)

Analysis of a numerical boundary layer

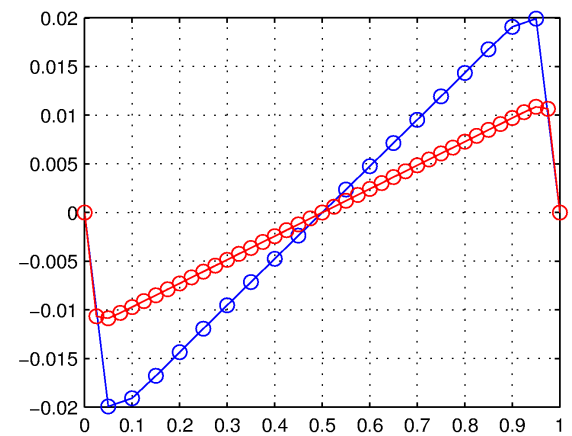
$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}u(x) = -4\pi^2\sin(2\pi x) \quad +\text{BCs} \qquad \longrightarrow u(x) = \sin(2\pi x)$$



- 5-point stencil discretizing $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$
- incompatibility at the boundary
- nearest neighbor extrapolation in ghost node



Error (periodic BCs)



Error (homog. Dirichlet BCs)

- Expansion of v requires order functions defined by purely discrete equations:

$$\Delta_h v = f \quad \text{approximate } u \text{ by} \quad v^{[n]} := \hat{u}^{(0)} + h\left(\hat{u}^{(1)} + s_h^{(1)}\right) + \ldots + h^n\left(\hat{u}^{(n)} + s_h^{(n)}\right)$$

- Standard stability estimate too crude $\longrightarrow$ damping property

$$\|\Delta_h^{-1}\|_\infty \, \|r_h^{[n]}\|_\infty \overset{h\to 0}{\not\longrightarrow} 0 \quad \text{but} \quad \|\Delta_h^{-1} r_h^{[n]}\|_\infty \overset{\epsilon\to 0}{\longrightarrow} 0$$

- D1P3 model $\Rightarrow$ no equivalent scalar equation!
- Regular asymptotic expansion:

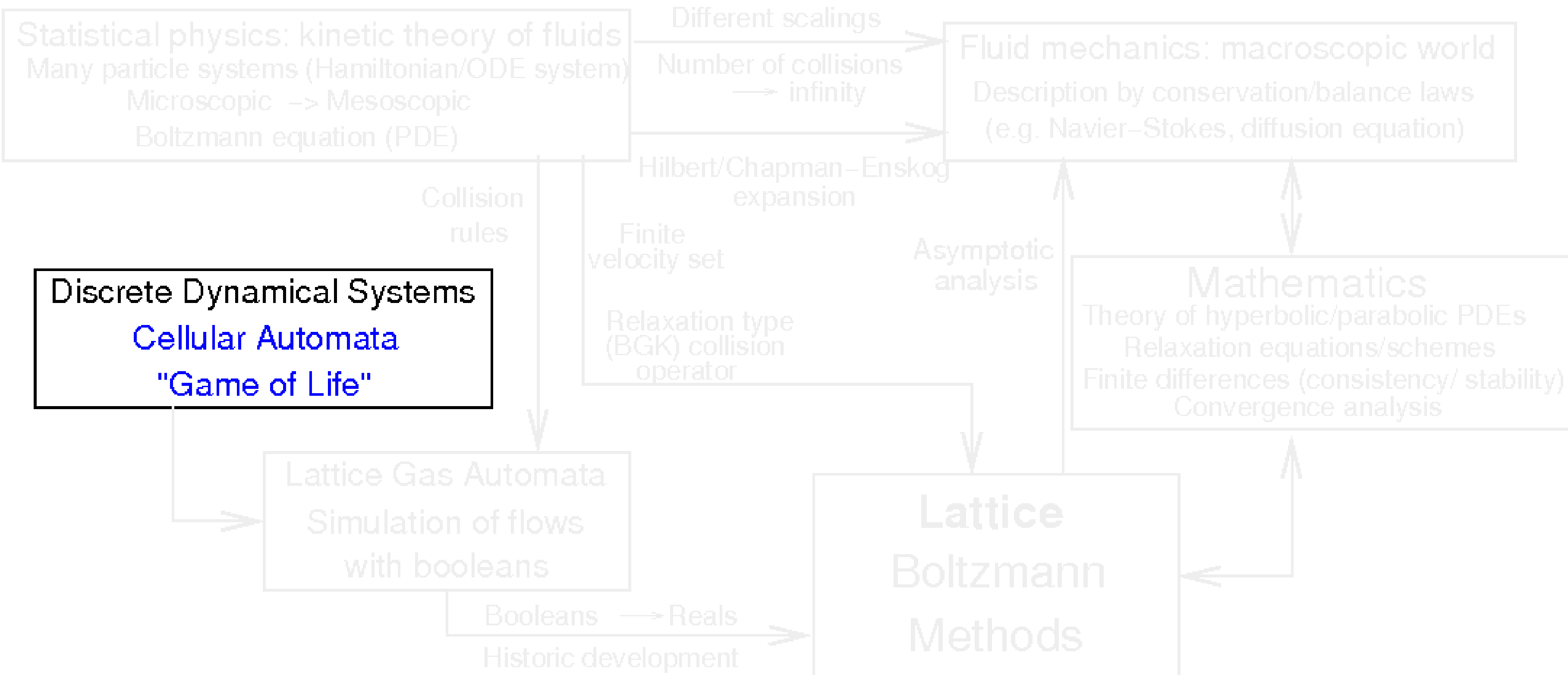
$$\mathsf{f} \approx \underbrace{\mathsf{f}^{(0)} + \epsilon \mathsf{f}^{(1)} + \epsilon^2 \mathsf{f}^{(2)} + \ldots + \epsilon^n \mathsf{f}^{(n)}}_{=:\mathsf{f}^{[n]}}$$

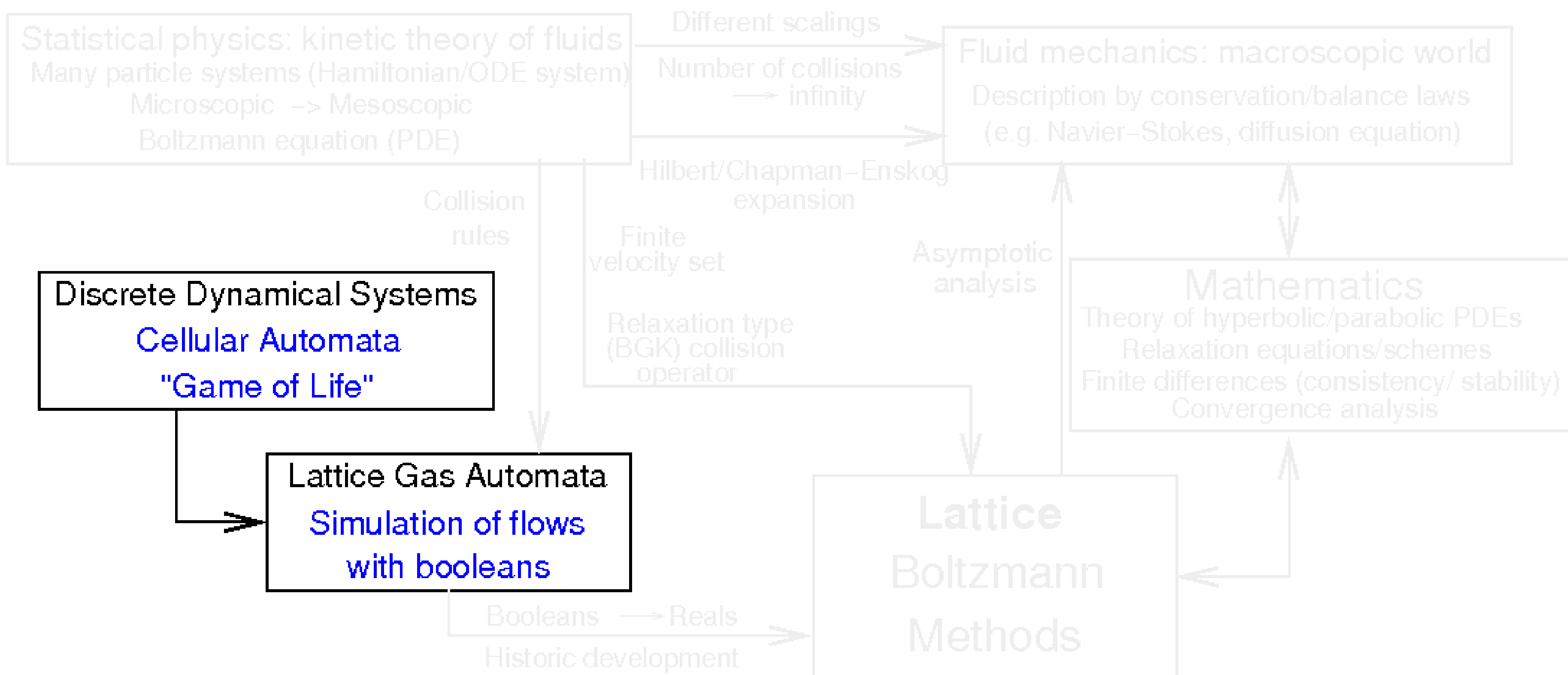
- **Residual:** $\partial_t \mathsf{f}^{[n]} + \epsilon^{-1} S \partial_x \mathsf{f}^{[n]} = \epsilon^{-2} J \mathsf{f}^{[n]} + \mathsf{r}^{[n]}$
- Determine $\mathsf{f}^{(0)}, \mathsf{f}^{(1)}, \ldots$ such that $\mathsf{r}^{[n]} \in \mathrm{O}(\epsilon^\alpha)$ with α as large as possible.
- $\mathsf{f} \leftrightarrow u$ with $\partial_t u - \frac{\tau}{6} \partial_x^2 u = 0$ $\quad \left\{ \begin{array}{lcl} \mathsf{f}^{(0)} & = & u\,\mathsf{w} \\ \mathsf{f}^{(1)} & = & -\tau \partial_x u\,\mathsf{sw} \\ \mathsf{f}^{(2)} & = & \tau^2 \partial_x^2 u\,(\mathsf{s}^2\mathsf{w} - \frac{1}{6}\mathsf{w}) \end{array} \right.$

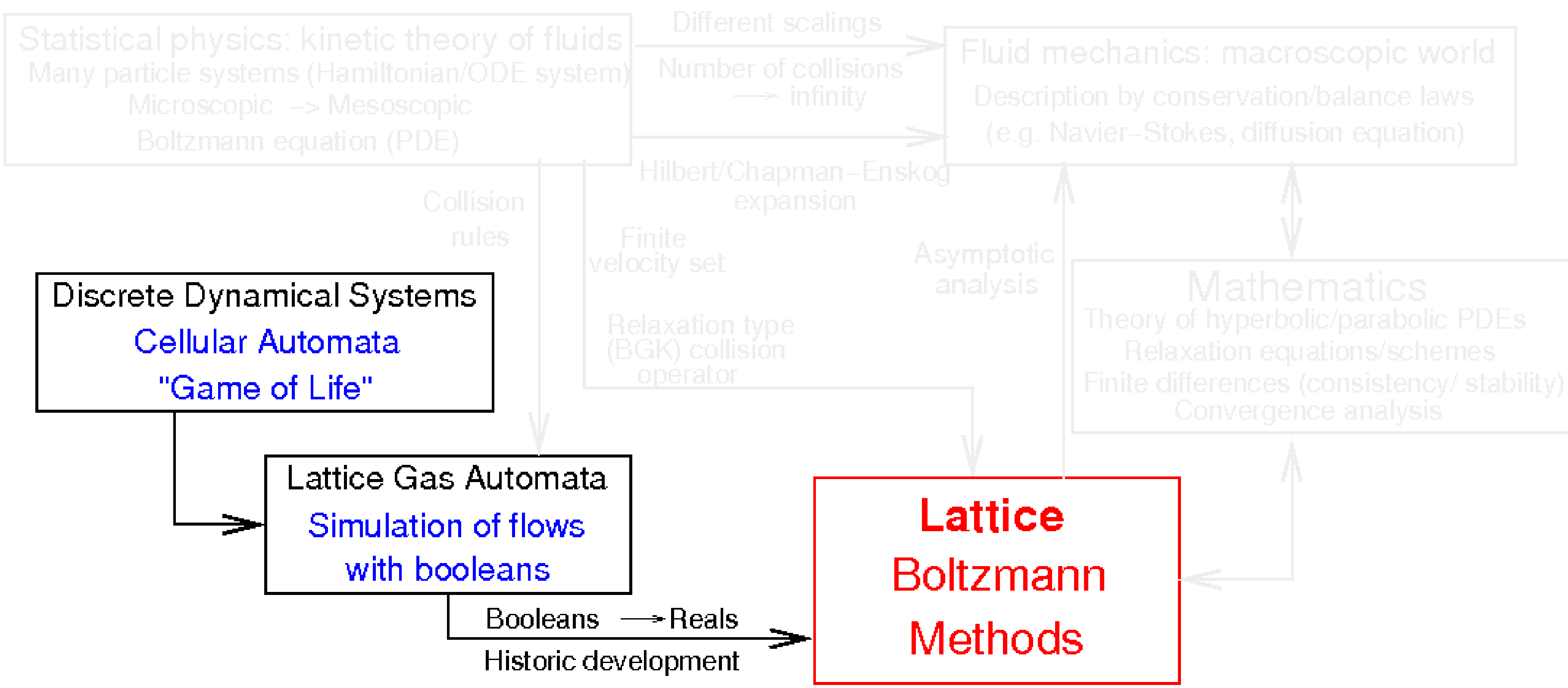
  $\Rightarrow$ consistency: $\langle \mathsf{f}, 1 \rangle = u$.
- Justification of regular expansion: consistency + stability $\Leftrightarrow$ convergence.
- **Theorem:** $\left\{ \begin{array}{ll} \mathsf{f}_\epsilon \in \mathcal{C}^1_{\mathrm{per}}(\mathcal{X}_T, \mathcal{F}) & \textit{solution of LBE} \\ \hat{\mathsf{f}}_\epsilon \in \mathcal{C}^1_{\mathrm{per}}(\mathcal{X}_T, \mathcal{F}) & \textit{approximate solution of LBE with residual} \in \mathrm{O}(\epsilon^\alpha) \end{array} \right.$

  $\|\mathsf{f}_\epsilon(0,\cdot) - \hat{\mathsf{f}}_\epsilon(0,\cdot)\|_{\mathcal{L}^2(\mathcal{X},\mathcal{F})} < K_0 \epsilon^\alpha$

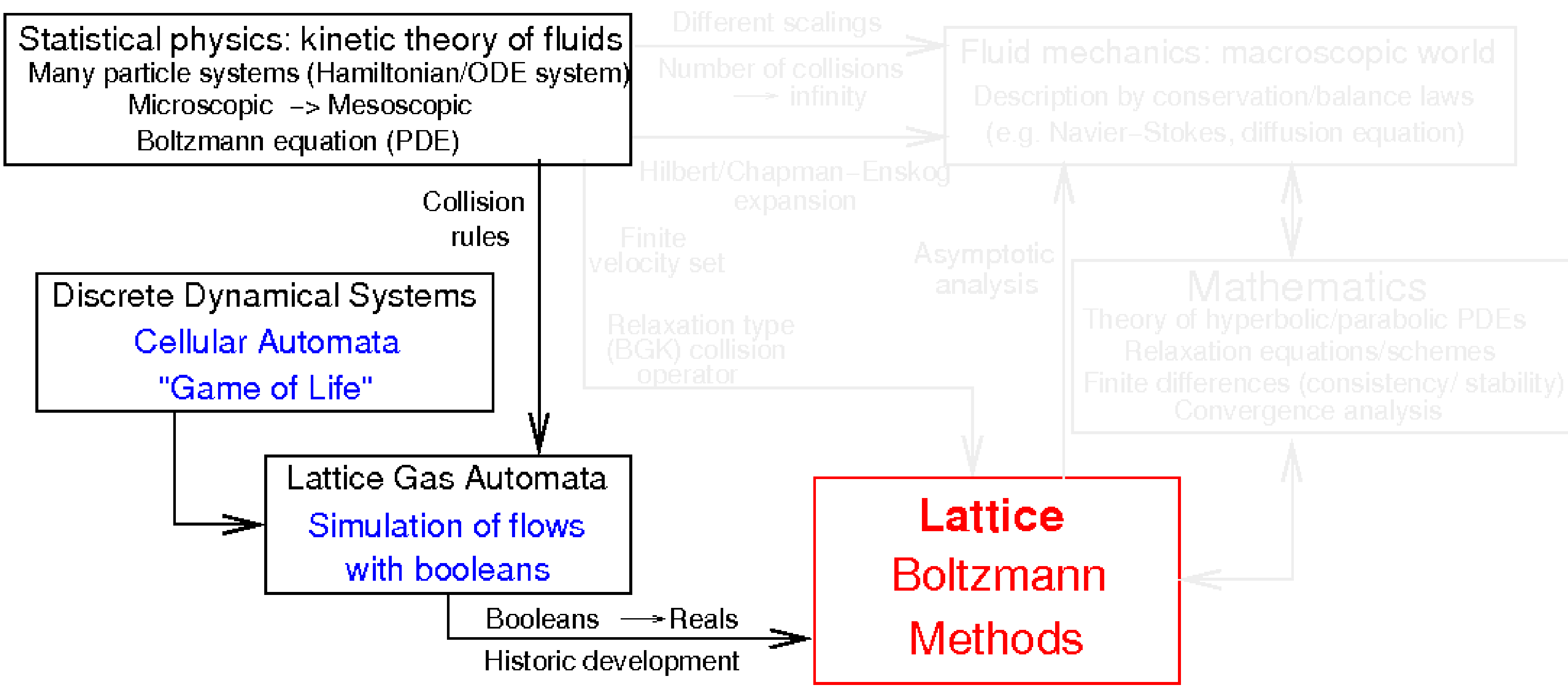
$$\Rightarrow \quad \sup_{t \in [0,T]} \|\mathsf{f}_\epsilon(t,\cdot) - \hat{\mathsf{f}}_\epsilon(t,\cdot)\|_{\mathcal{L}^2(\mathcal{X},\mathcal{F})} < K_0 \epsilon^\alpha$$

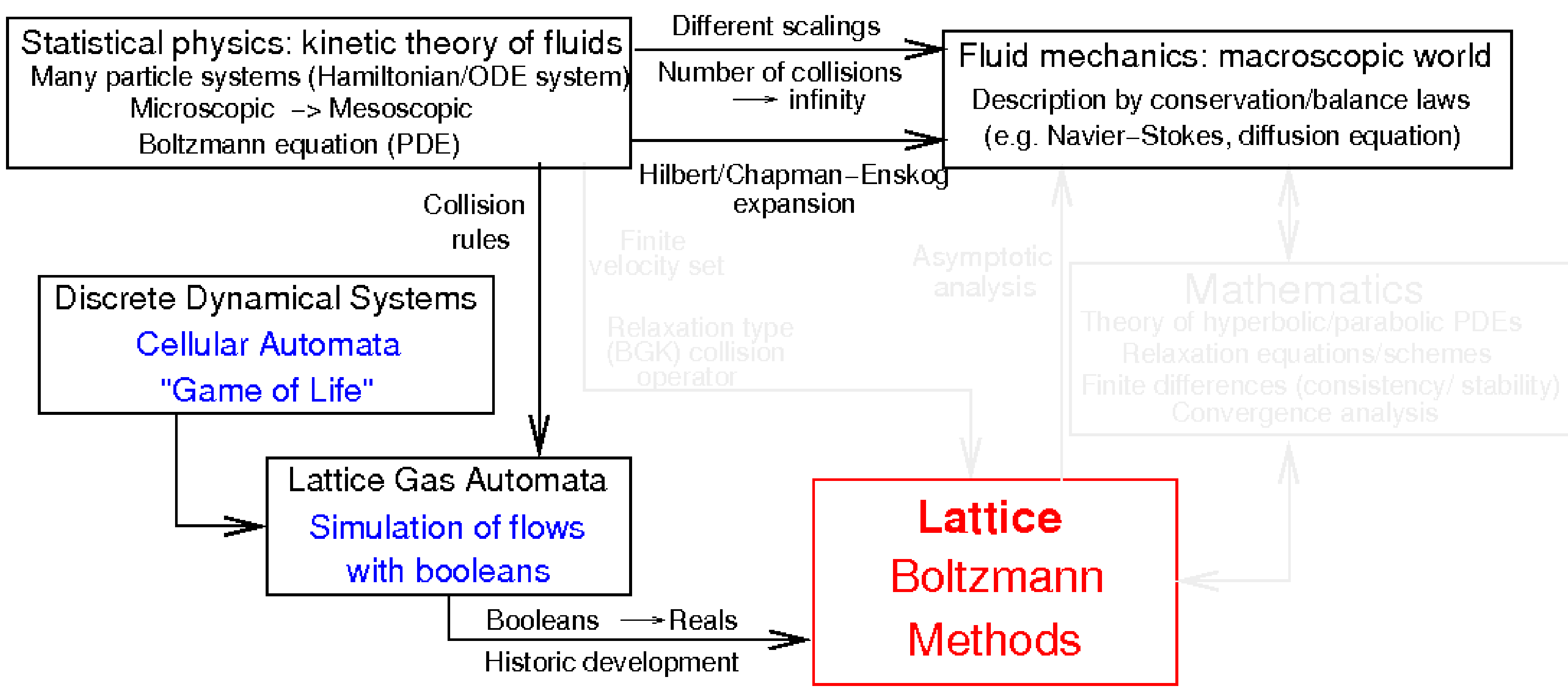

Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions → infinity
Hilbert/Chapman–Enskog expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier–Stokes, diffusion equation)
Collision rules
Finite velocity set
Relaxation type (BGK) collision operator
Asymptotic analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Lattice Gas Automata
Simulation of flows
with booleans
Booleans → Reals
Historic development
Lattice
Boltzmann
Methods

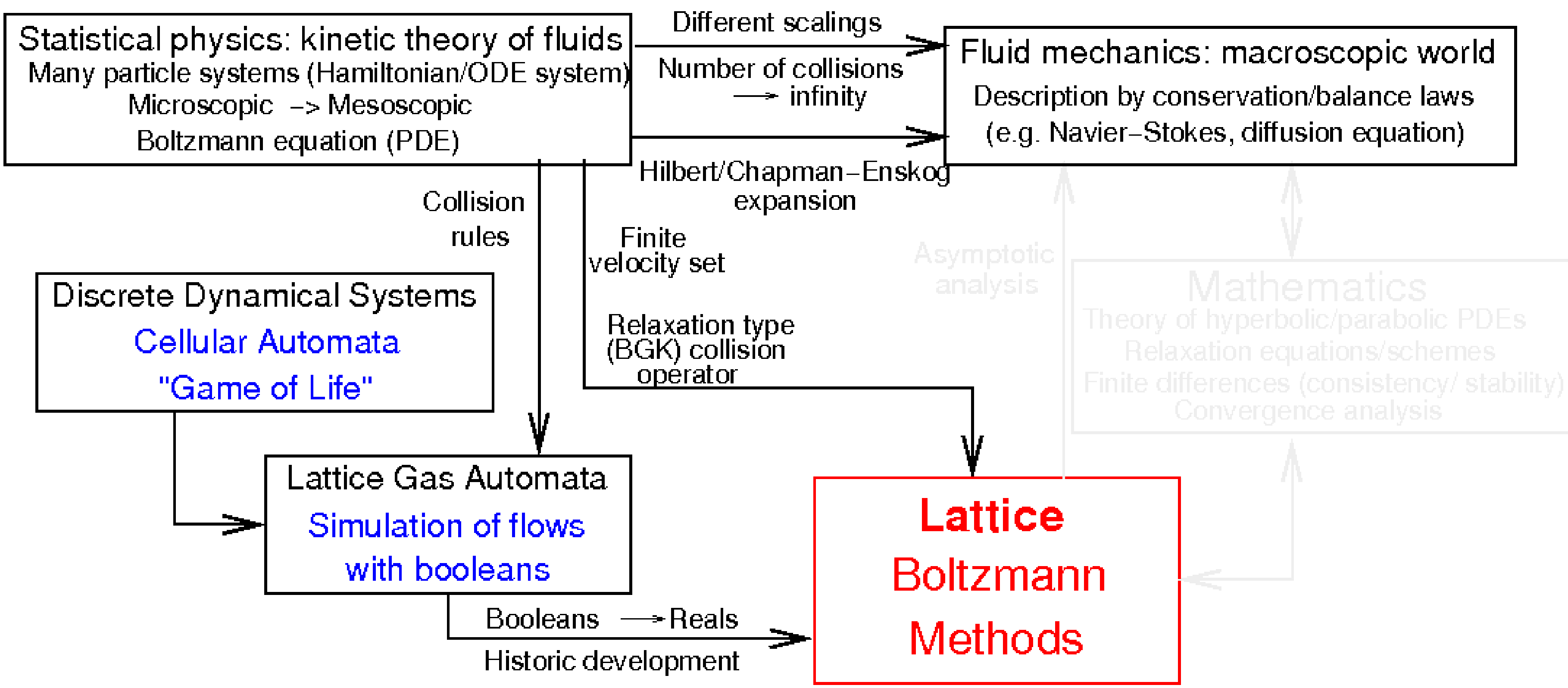

Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic –> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions → infinity
Hilbert/Chapman–Enskog expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier–Stokes, diffusion equation)
Collision rules
Finite velocity set
Relaxation type (BGK) collision operator
Asymptotic analysis
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Lattice Gas Automata
Simulation of flows with booleans
Booleans → Reals
Historic development
Lattice Boltzmann Methods

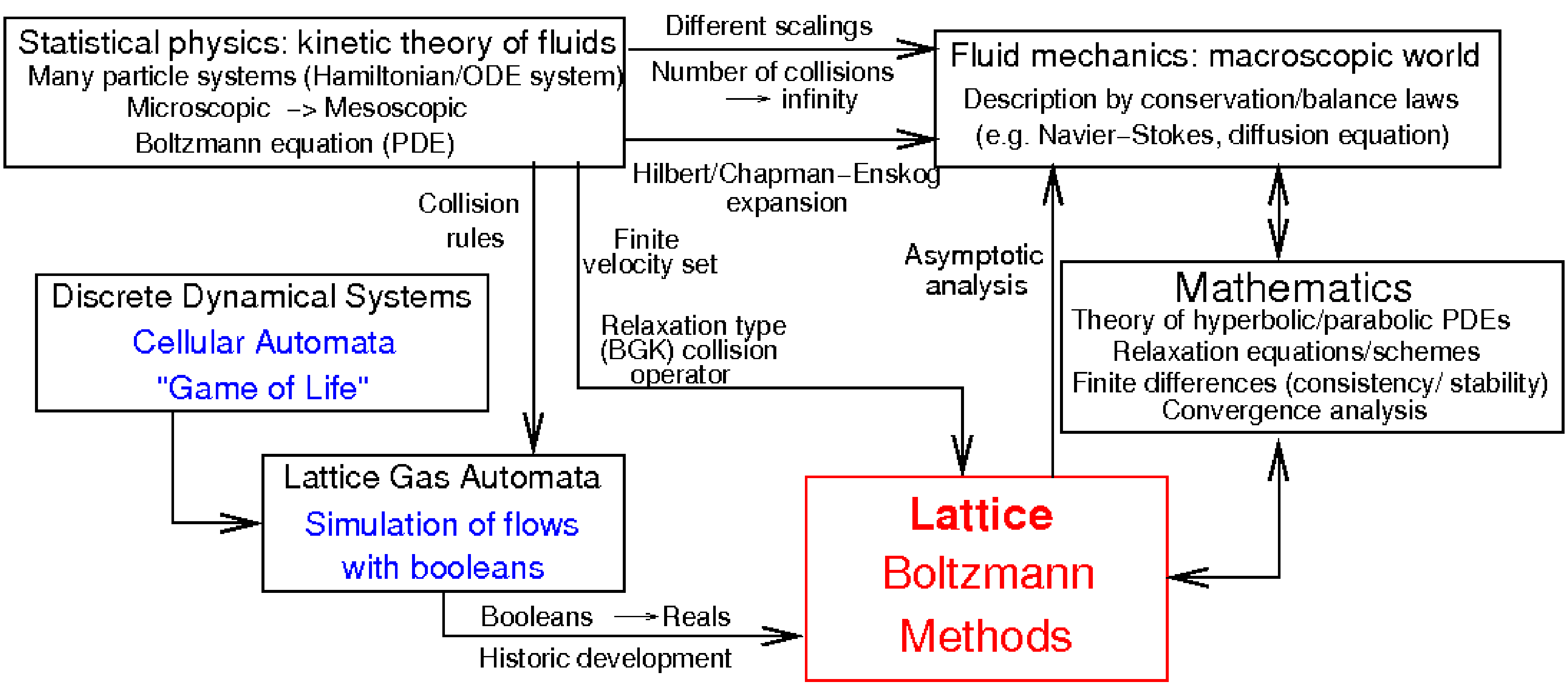


Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions → infinity
Hilbert/Chapman–Enskog expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier–Stokes, diffusion equation)
Collision rules
Finite velocity set
Relaxation type (BGK) collision operator
Asymptotic analysis
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Lattice Gas Automata
Simulation of flows with booleans
Booleans → Reals
Historic development
Lattice Boltzmann Methods


Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions → infinity
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier–Stokes, diffusion equation)
Hilbert/Chapman–Enskog expansion
Collision rules
Finite velocity set
Asymptotic analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Relaxation type (BGK) collision operator
Lattice Gas Automata
Simulation of flows with booleans
Lattice Boltzmann Methods
Booleans → Reals
Historic development


Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions
→ infinity
Hilbert/Chapman−Enskog
expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier−Stokes, diffusion equation)
Collision
rules
Finite
velocity set
Relaxation type
(BGK) collision
operator
Asymptotic
analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Lattice Gas Automata
Simulation of flows
with booleans
Booleans → Reals
Historic development
Lattice
Boltzmann
Methods


Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions
→ infinity
Hilbert/Chapman−Enskog
expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier−Stokes, diffusion equation)
Collision
rules
Finite
velocity set
Relaxation type
(BGK) collision
operator
Asymptotic
analysis
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Lattice Gas Automata
Simulation of flows
with booleans
Booleans → Reals
Historic development
Lattice
Boltzmann
Methods


Statistical physics: kinetic theory of fluids
Many particle systems (Hamiltonian/ODE system)
Microscopic -> Mesoscopic
Boltzmann equation (PDE)
Different scalings
Number of collisions → infinity
Hilbert/Chapman–Enskog expansion
Fluid mechanics: macroscopic world
Description by conservation/balance laws
(e.g. Navier–Stokes, diffusion equation)
Collision rules
Finite velocity set
Relaxation type (BGK) collision operator
Asymptotic analysis
Discrete Dynamical Systems
Cellular Automata
"Game of Life"
Mathematics
Theory of hyperbolic/parabolic PDEs
Relaxation equations/schemes
Finite differences (consistency/ stability)
Convergence analysis
Lattice Gas Automata
Simulation of flows with booleans
Booleans → Reals
Historic development
Lattice Boltzmann Methods