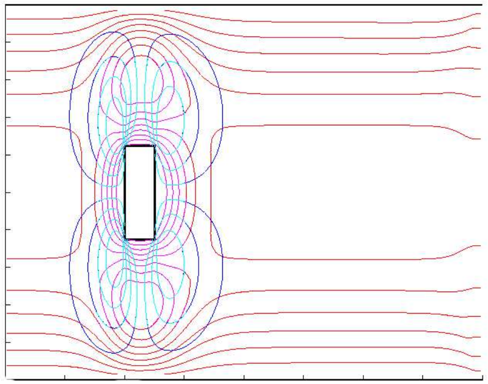
Asymptotic Investigation of the Lattice-Boltzmann Method and Grid Coupling



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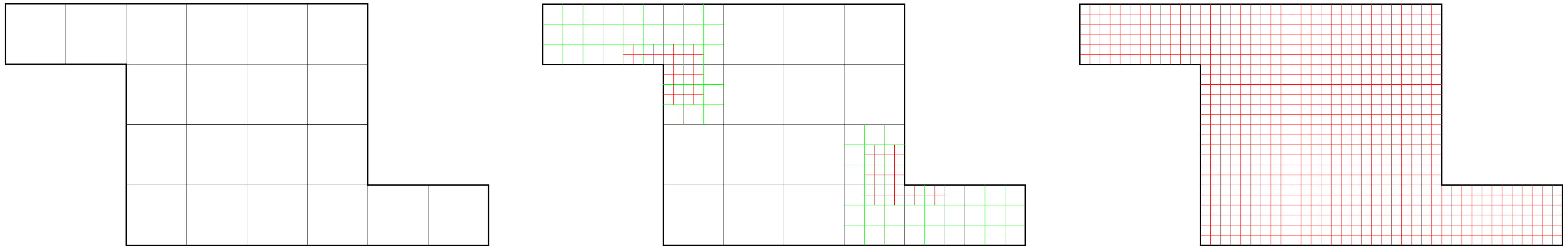
HYKE Workshop, Uni Saarbrücken
February 23-25, 2004

Introduction

Project: *Consistent* grid coupling algorithms for LB methods

Motivation: A priori grid refinement

Problem: Standard LB-algorithm: only on uniform grids !

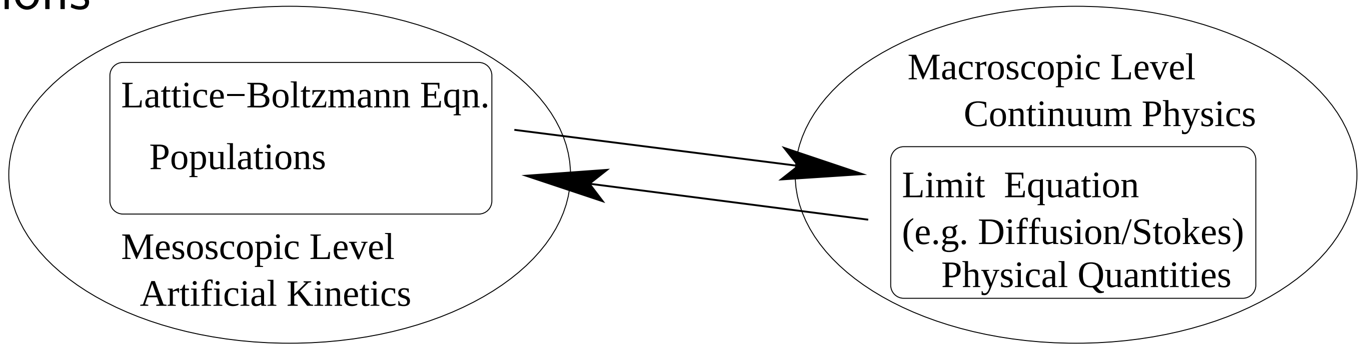


Strategy:
– understand simple cases
– generalize to complicated problems

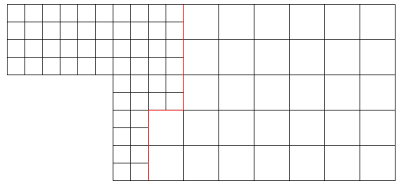
Macroscopic coupling conditions

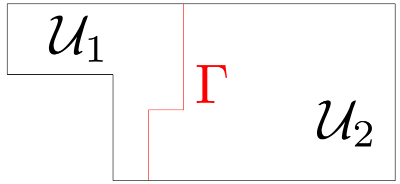
Analysis of the LB-method

Synthesis $\rightarrow$ LB coupling



Domain Decomposition





A paradigm: heat equation $\partial_t v - \nu \Delta v = g$

Idea: Solve eqn. in $\mathcal{U}_1, \mathcal{U}_2$ separately: v_1, v_2

Question: $v = \tilde{v}$? $\quad \tilde{v}(\cdot, \mathbf{x}) := \begin{cases} v_1(\cdot, \mathbf{x}), & \mathbf{x} \in \mathcal{U}_1 \\ v_2(\cdot, \mathbf{x}), & \mathbf{x} \in \mathcal{U}_2 \end{cases}$

Remark: Any *smooth* solution is a *weak* solution.
v_1, v_2 smooth: When $\tilde{v}$ is a *global weak* solution, i.e. $\forall \phi \in \mathcal{C}_c^\infty(\mathcal{I} \times \mathcal{U})$:

$$-\int_{\mathcal{I}\times\mathcal{U}} \left(\tilde{v}\partial_t \phi + \nu \nabla \tilde{v} \cdot \nabla \phi \right) \overset{!}{=} \sum_{i=1}^{2} \int_{\mathcal{I}\times\mathcal{U}_i} \left(\partial_t v_i - \nu \Delta v_i \right) \phi \; = \int_{\mathcal{I}\times\mathcal{U}} g\, \phi$$

$\Leftrightarrow$ **Interface conditions:**

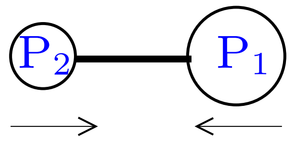
$$v_1\,|_\Gamma \overset{!}{=} v_2\,|_\Gamma \quad \wedge \quad \mathbf{n} \cdot \nabla v_1\,|_\Gamma \overset{!}{=} \mathbf{n} \cdot \nabla v_2\,|_\Gamma$$

Physical meaning: equality of v and its *normal flux* (heat flow through Γ)

1D model problem: 

Coupling conditions: $v_1(t, \xi) = v_2(t, \xi) \quad \wedge \quad \partial_x v_1(t, \xi) = \partial_x v_2(t, \xi)$

D1P2-Model for the Heat Equation

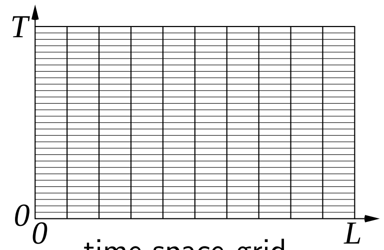


Velocity space : $\mathcal{S} = \{\mathrm{s}_1, \mathrm{s}_2\} = \{-1, 1\}$

Grid spacing : $h \in \mathcal{H} = \{\frac{L}{M} : M \in \mathbb{N}\}$

Primary variables : $\mathrm{P} : \mathcal{T}(h^2) \times \mathcal{G}(h) \times \mathcal{S} \to \mathbb{R}$

Density : $U = \mathrm{P}_1 + \mathrm{P}_2$



time-space grid

Discrete LBE with diffusive scaling:

$$\mathrm{P}(t+h^2, x+\mathrm{s}\,h, \mathrm{s}) - \mathrm{P}(t, x, \mathrm{s}) = \frac{1}{\tau}\Big(\frac{1}{2}U(t,x) - \mathrm{P}(t,x,\mathrm{s})\Big) + \frac{1}{2}h^2\,g(t,x)$$

Remark: $U \approx v\big|_{\mathcal{T}(h^2)\times\mathcal{G}(h)}$ with $\partial_t v - \nu\partial_x^2 v = g \qquad \nu = \tau - \frac{1}{2}$

Working hypothesis:

$\exists$ smooth functions $\mathrm{p}^{(l)} : \mathbb{R}_0^+ \times [0, L] \times \mathcal{S} \to \mathbb{R} \quad 0 \le l \le 4$

$\exists$ bounded grid function $R : \mathcal{H} \times \mathcal{T}(h^2) \times \mathcal{G}(h) \times \mathcal{S} \to \mathbb{R}$

$$\mathrm{P} = \mathrm{p}^{(0)} + h\,\mathrm{p}^{(1)} + \ldots + h^4\,\mathrm{p}^{(4)} + h^5\,R \qquad \text{(ansatz)}$$

Formal Asymptotic Analysis

- $u^{(l)} := \mathrm{p}_1^{(l)} + \mathrm{p}^{(l)}{}_2$, plug ansatz into discrete LBE:
  assume equation valid $\forall (t,x) \in \mathbb{R}_0^+ \times [0,L], \;\; h > 0$

- Finite difference $\longrightarrow$ Taylor

$$\sum_{l=0}^{4} h^l \left[h^2 \partial_t \mathrm{p}^{(l)} + h\,\mathrm{s}\,\partial_x \mathrm{p}^{(l)} + \frac{1}{2} h^2 \partial_x^2\,\mathrm{p}^{(l)} \right] + O(h^3) \;=\; \frac{1}{\tau} \sum_{l=0}^{4} h^l \left[\frac{1}{2} u^{(l)} - \mathrm{p}^{(l)} \right] + \frac{1}{2} h^2 g$$

- Collect terms of equal order in h

$$\begin{aligned} h^0 :\; \mathrm{p}^{(0)} &= \tfrac{1}{2} u^{(0)} \\ h^1 :\; \mathrm{p}^{(1)} &= \tfrac{1}{2} u^{(1)} - \tau\,\mathrm{s}\partial_x \mathrm{p}^{(0)} \\ h^2 :\; \mathrm{p}^{(2)} &= \tfrac{1}{2} u^{(2)} - \tau\,\mathrm{s}\partial_x \mathrm{p}^{(1)} - \tfrac{1}{2}\tau \partial_x^2 \mathrm{p}^{(0)} - \tau \partial_t \mathrm{p}^{(0)} + \tfrac{1}{2}\tau g \quad \text{sum over } \mathrm{s} \end{aligned}$$

$$\Rightarrow \quad \partial_t\, u^{(0)} \;-\; \left(\tau - \tfrac{1}{2}\right)\, \partial_x^2\, u^{(0)} \;=\; g$$

- **Observation:** Since $u^{(0)} = v$: $\quad U = v + h\, u^{(1)} + O(h^2)$

Formal Asymptotic Analysis (cont.)

- 3^{rd} order: $\partial_t\, u^{(1)} \;-\; (\tau - \frac{1}{2})\, \partial_x^2\, u^{(1)} \;=\; 0$
- Therefore: $u^{(1)} \equiv 0$ if $U(0,\cdot) = v(0,\cdot) + 0\,h + \ldots.$
- 4^{th} order: $\partial_t\, u^{(2)} - (\tau - \frac{1}{2})\, \partial_x^2\, u^{(2)} \;=\; \mathrm{RHS}$

  $\mathrm{RHS} = -(\tau^3 - 2\tau^2 + \frac{2}{3}\tau - \frac{1}{3})\, \partial_x^4\, u^{(0)} - (\tau^2 - \frac{3}{2}\tau + \frac{1}{4})\, \partial_x^2\, g - \frac{1}{2}\, \partial_t\, g$

- **Observation:** Discrete LBE: 2^{nd} order consistent to heat equation.

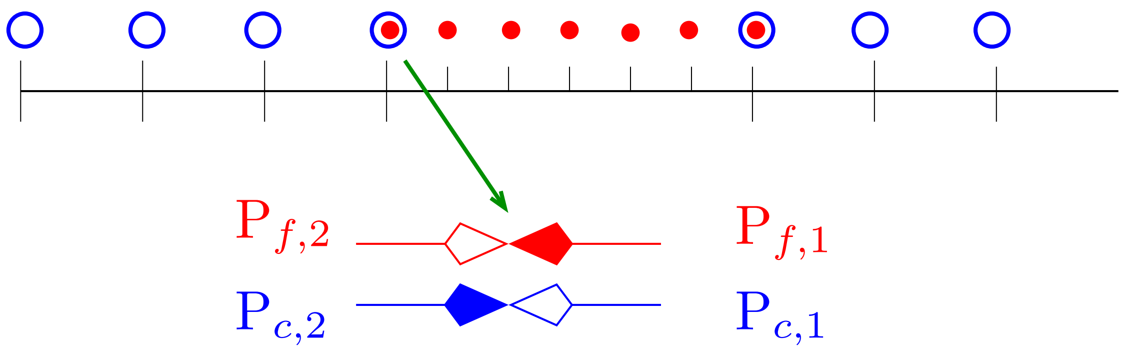
$$U = v + h^2\, u^{(2)} + O(h^3)$$

- Relation between v and $\mathrm{P} = \mathrm{p}^{(0)} + h\, \mathrm{p}^{(1)} + h^2\, \mathrm{p}^{(2)} + O(h^3)$:

$$\begin{aligned}
\mathrm{p}^{(0)} &= \tfrac{1}{2} v \\
\mathrm{p}^{(1)} &= -\tfrac{1}{2}\tau\, \mathrm{s}\, \partial_x\, v \\
\mathrm{p}^{(2)} &= u^{(2)} \\
\mathrm{p}^{(3)} &= u^{(3)} - \tfrac{1}{2}\,\tau\, \mathrm{s}\, \partial_x u^{(2)} + \tfrac{1}{2}\,\tau\,(\tau - 1)\, \mathrm{s}\, \partial_x\, \partial_t\, v + \tfrac{1}{12}\,\tau\, \mathrm{s}\,(3\tau - 1)\, \partial_x^3\, v
\end{aligned}$$

- $F \;:=\; h^{-1}\,(\mathrm{P}_2 - \mathrm{P}_1) \;=\; -\tau\, \partial_x\, v + O(h^2)$

Synthesis of the Coupling Condition



known: $P_{c,2}$, $P_{f,1}$

empty: $P_{c,1}$, $P_{f,2}$

- Translate: *macroscopic condition* $\longrightarrow$ *mesoscopic LB level*
- Refinement factor: $N \in \mathbb{N}$: $h_f = h$, $h_c = N\,h$

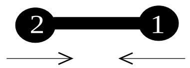
$$\begin{array}{rcl} v_{\text{left}} \overset{!}{=} v_{\text{right}} & \Rightarrow & P_{c,1} + P_{c,2} = P_{f,1} + P_{f,2} \\ \partial_x v_{\text{left}} \overset{!}{=} \partial_x v_{\text{right}} & \Rightarrow & P_{c,1} - P_{c,2} = N\,(P_{f,1} - P_{f,2}) \end{array} \qquad (*)$$

- solve for empty pops: $P_{c,1}$, $P_{f,2}$

Analysis of the Coupling Condition:

– Separate asymptotic *ansatz* for coarse and fine grid with h_c, h_f as above
– Plug into $(*)$ $\longrightarrow$ equate terms of equal order w.r.t. h
– Extract interface conditions for $u^{(0)}, u^{(1)}, u^{(2)}$

The Coupling Algorithm



global TimeStep:

collide & propagate on coarse-grid

interpolate known coarse-grid interface-pops

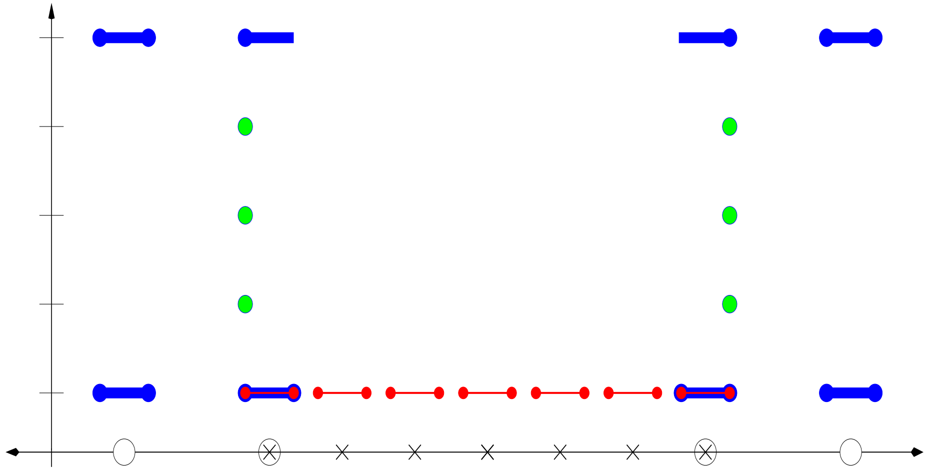
repeat N^2 times

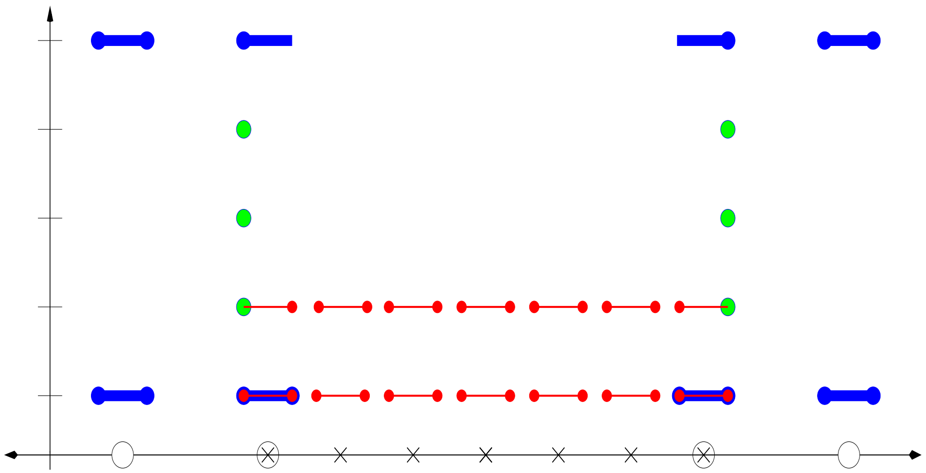
  collide & propagate on fine-grid

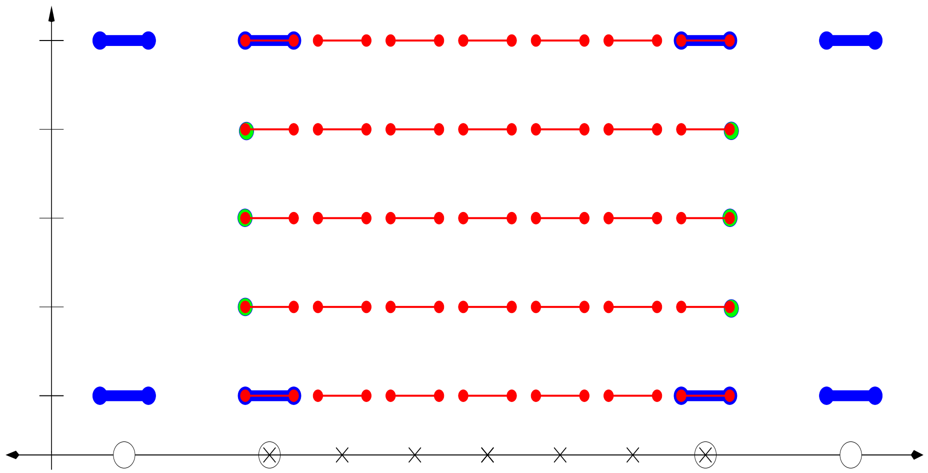
  fill empty fine-grid interface-pops

end

fill empty coarse-grid interface-pops







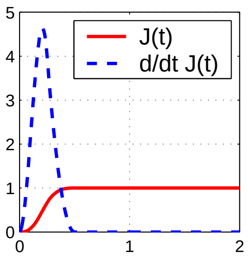
D1P2 Model: Numeric Test - Snapshot $\mathbf{t = 0.8}$

Example: $\nu = 0.001 \quad \Rightarrow \quad \tau = 0.501$ domain: $\mathbb{T}^1$

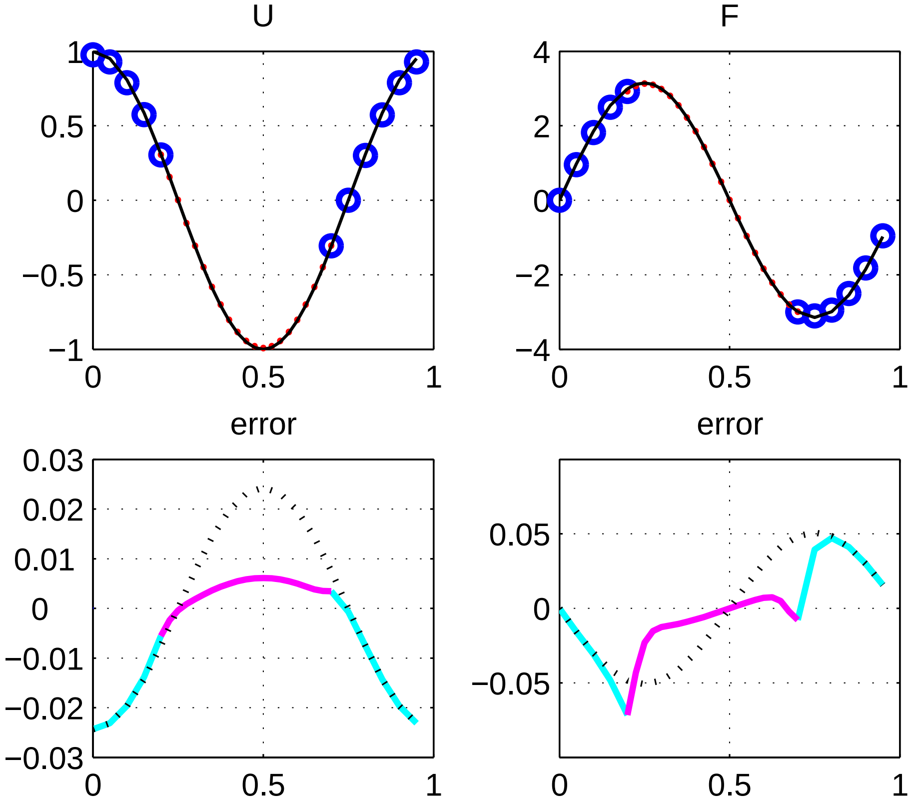
$$v(t,x) = J(t)\ \cos(2\pi\, x), \qquad f := -\tau\, \partial_x\, v(t,x) = J(t)\, 2\pi\tau \sin(2\pi x)$$

$$g(t,x) = J(t)\, 4\pi^2\nu \cos(2\pi x)\ +\ J'(t)\ \cos(2\pi x)$$

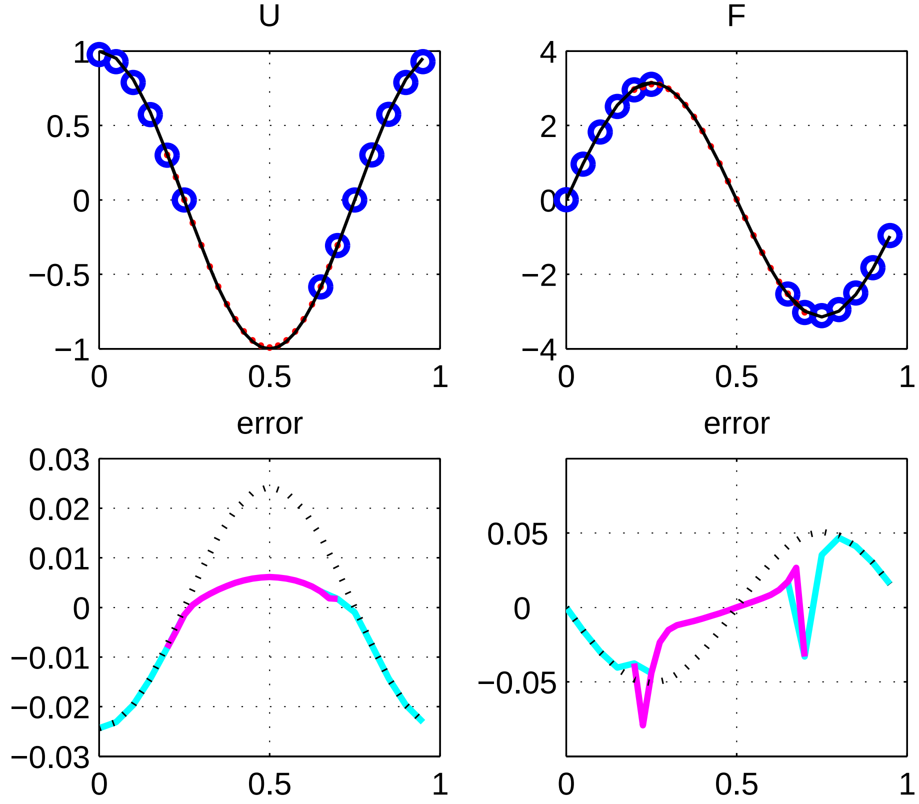
Uniform coarse grid: 20 nodes, interface nodes: 5 (left), 15 (right), $N = 2$



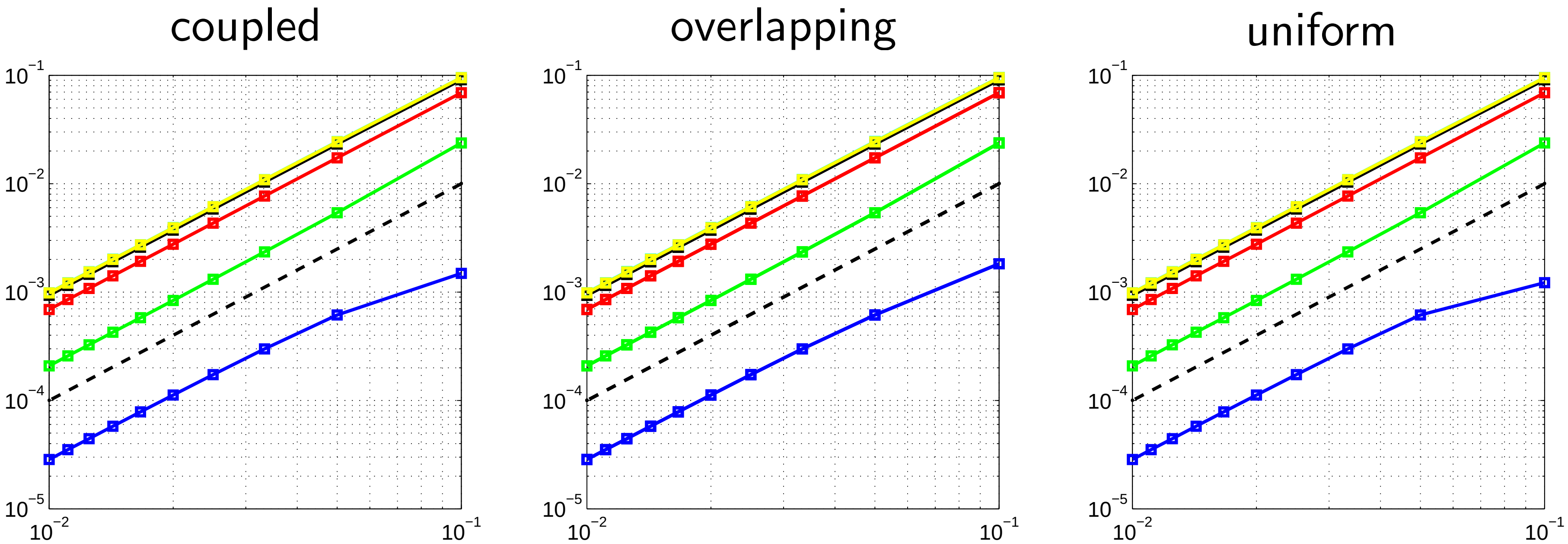
Coupled Grids



Overlapping Grids



D1P2 Model: Numeric Test - L^∞ Error of U versus ds

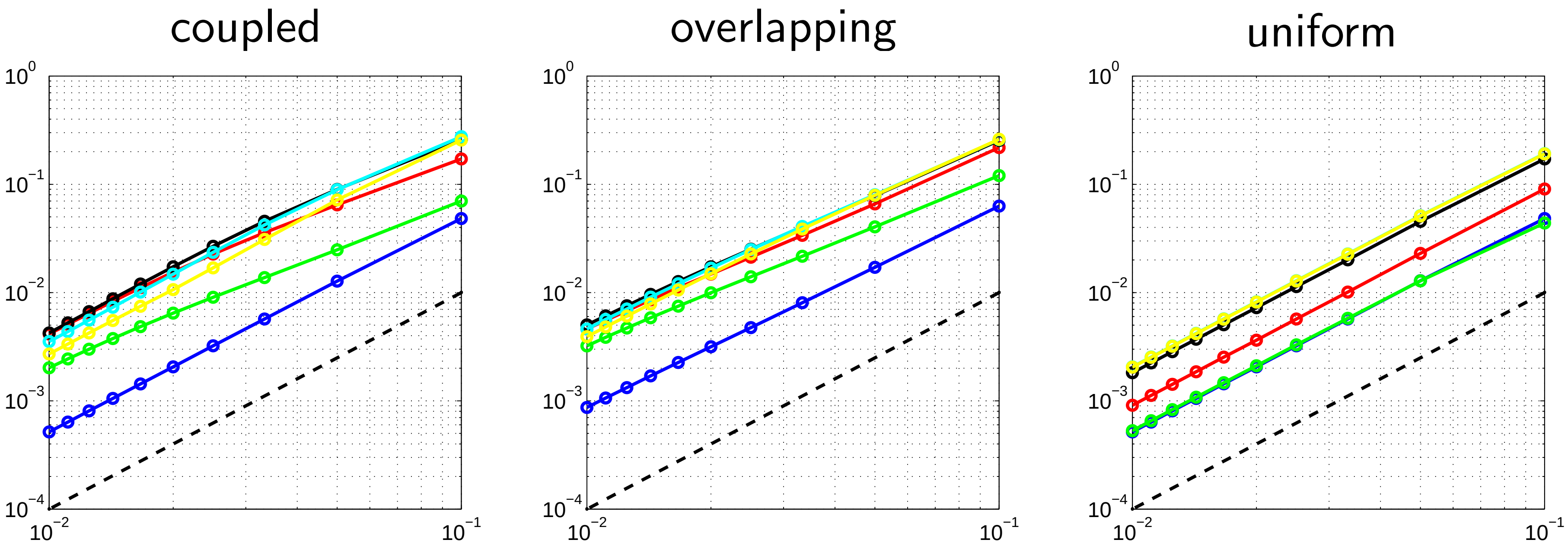


Convergence rates:
(Linear least square fit)

	time	coupled	overlapping	uniform
–	0.1	1.769	1.832	1.706
–	0.2	2.048	2.048	2.048
–	0.3	2.000	2.000	2.000
–	0.4	1.991	1.991	1.991
–	0.5	1.988	1.988	1.988
–	0.8	1.988	1.988	1.988

Setting: nu=0.001, M=10, West=3, East=8, N=2, maxIt=200, Step=10, InitMode=1, InitPhase=0.5, Frequency=1, smoothStart with source

D1P2 Model: Numeric Test - L^∞ Error of F versus ds



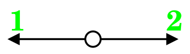
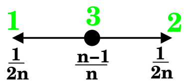
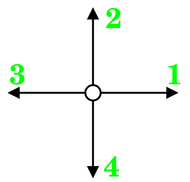
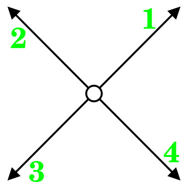
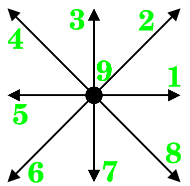
Convergence rates:

(Linear least square fit)

	time	coupled	overlapping	uniform
–	0.1	1.977	1.853	1.978
–	0.2	1.530	1.563	1.934
–	0.3	1.627	1.661	2.001
–	0.4	1.812	1.696	1.981
–	0.5	1.926	1.747	1.977
–	0.8	1.995	1.831	1.977

Setting: nu=0.001, M=10, West=3, East=8, N=2, maxIt=200, Step=10,
InitMode=1, InitPhase=0.5,Frequency=1, smoothStart with source

LB-Models

	D1P2	*DAR 1D*	$\nu = \tau$	$Q(u, \mathrm{s}) \;=\; \frac{1}{2} u\,(1 + \epsilon\, \mathrm{s} a - \epsilon^2\, \tau\, c)$	$\mathrm{k} \;=\; \frac{1}{2}\, g$
	D1P3	''	$\nu = \frac{\tau}{n}$	$Q(u, \mathrm{s}) \;=\; \mathrm{w_s} u\,(1 + \epsilon\, n\, \mathrm{s}\, a - \epsilon^2\, \tau\, c)$	$\mathrm{k} \;=\; \mathrm{w_s}\, g$
	D2P4	*DAR 2D*	$\nu = \frac{\tau}{2}$	$Q(u, \mathbf{s}) \;=\; \frac{1}{4} u\,(1 + 2\epsilon\, \mathbf{s} \cdot \mathbf{a} - \epsilon^2\, \tau\, c)$	$\mathrm{k} \;=\; \frac{1}{4}\, g$
	D2P4X	''	$\nu = \tau$	$Q(u, \mathbf{s}) \;=\; \frac{1}{4} u\,(1 + \epsilon\, \mathbf{s} \cdot \mathbf{a} - \epsilon^2\, \tau\, c)$	$\mathrm{k} \;=\; \frac{1}{4}\, g$
	D2P9	*Stokes-Oseen*	$\nu = \frac{\tau}{3}$	$Q(p, \mathbf{u}, \mathbf{s}) = \; 3\, \mathrm{w_s}(p + \mathbf{s} \cdot \mathbf{u}) \quad \ldots$ $+\, 9 \mathrm{w_s}\, \epsilon\, \big[3(\mathbf{s} \cdot \mathbf{a})\,(\mathbf{s} \cdot \mathbf{u}) - \mathbf{a} \cdot \mathbf{u}\big]$	$\mathrm{k} \;=\; 3 \mathrm{w_s}\, \mathbf{s} \cdot \mathbf{g}$

Limit equations:

i) $\partial_t u \;+\; \partial_x(a u - \nu \partial_x u) \;+\; c u \;=\; g$ (*DAR 1D*)

ii) $\partial_t u \;+\; \nabla \cdot (\mathbf{a} u - \nu \nabla u) \;+\; c u \;=\; g$ (*DAR 2D*)

iii) $\partial_t \mathbf{u} \;+\; \mathbf{a} \cdot \nabla \mathbf{u} \;-\; \nu \Delta \mathbf{u} \;=\; -\nabla p + \mathbf{g} \qquad \nabla \cdot \mathbf{u} = 0$

Remark: 2D LB-models reduce to 1D LB-models in case of translational invariance.

Coupling for other LB Models

D1P2:

$\#\{$macroscopic coupling cond.$\} = \#\{$empty interface pops$\} = \#\{$pops per node$\}$

Challenges:

i) $\#\{$*macroscopic* coupling conditions$\} < \#\{$*empty* interface pops$\}$
too much LB interface-conditions $\rightarrow$ contradictive w.r.t. macroscopic level

ii) 2D: hanging nodes $\rightarrow$ spatial interpolation

iii) 2D: corner nodes $\leftarrow$ boundary not smooth (!)

iv) Generally: initial layers $\Rightarrow$ working assumption violated

Possible problems:

i) loss of 2^{nd} order consistency for moments of 1^{st} and higher order

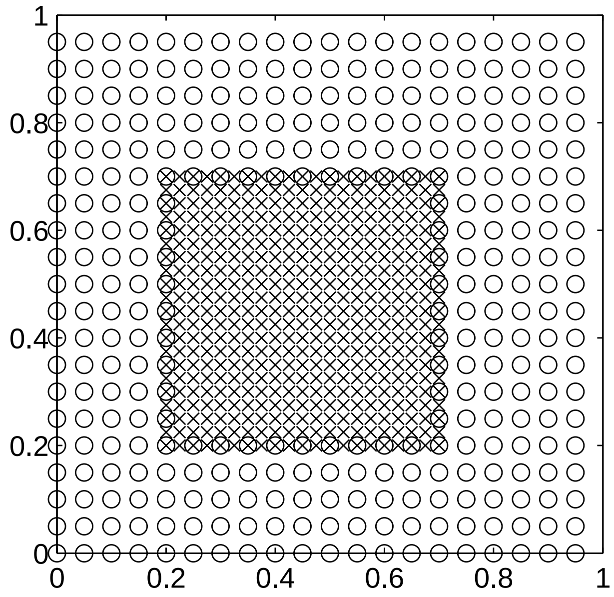
ii) reduction of stability

D2P9 Model: Numeric Test - Snapshot $t = 0.5$

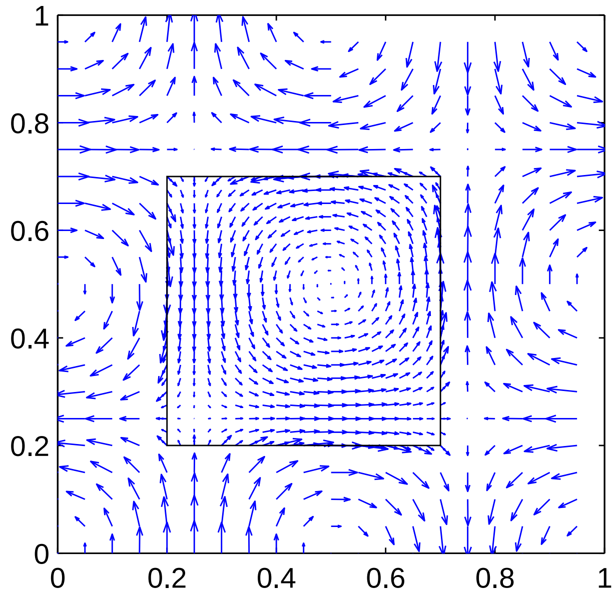
Benchmark: domain $\mathbb{T}^2$, $\nu = 0.01 \quad \tau = 0.53$

Stationary eigenmode of Stokes operator with smooth initialization $(\mathbf{u} = J(t)\mathbf{e})$

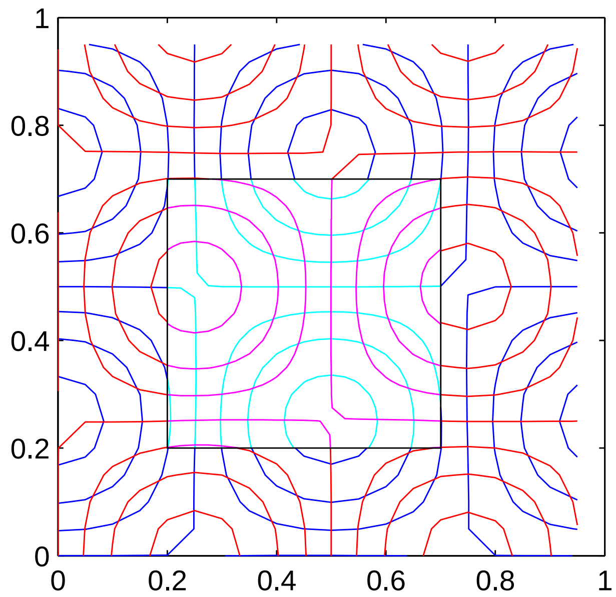
Partially Refined Grid

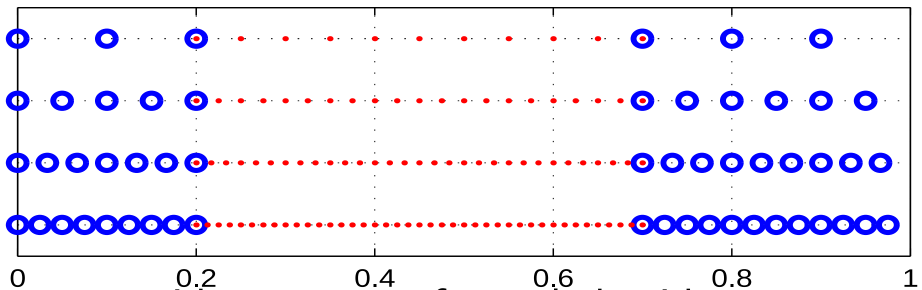


Velocity Field

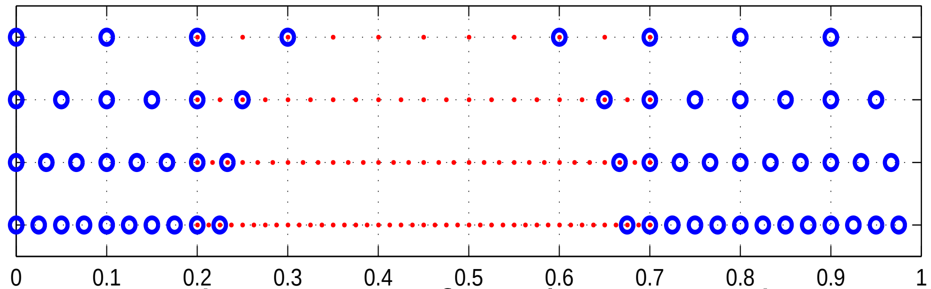


U_x, U_y Contour Lines



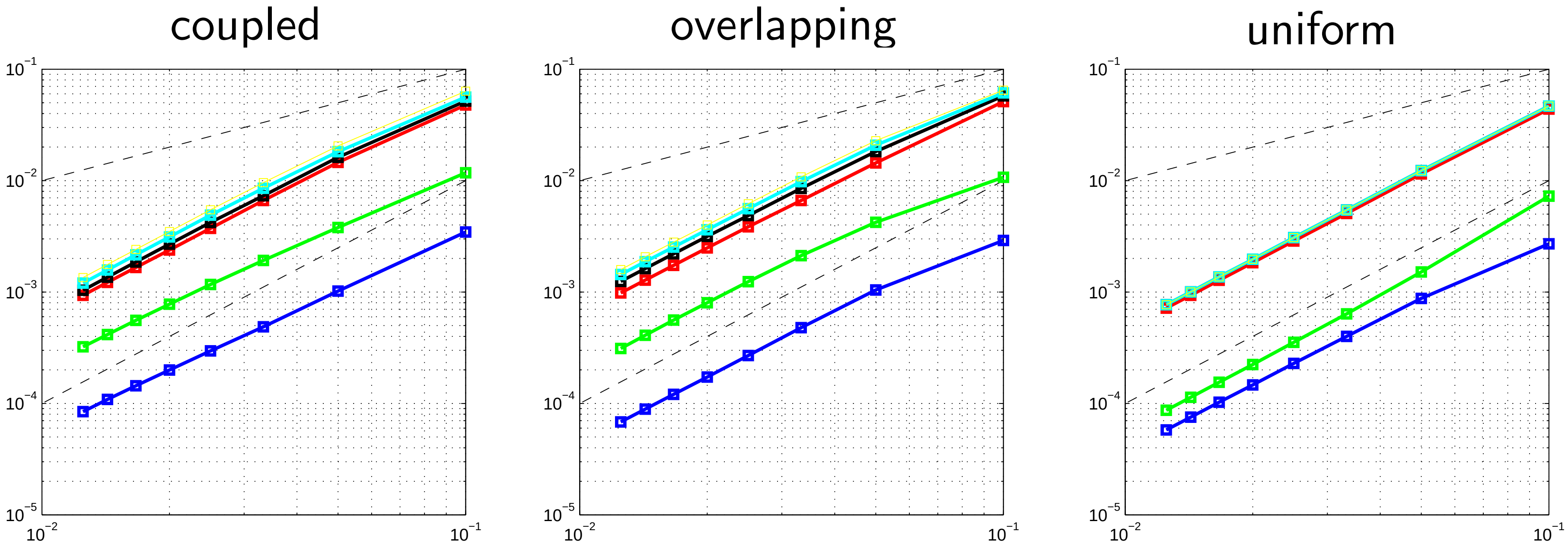


grid sequence of coupled grids



grid sequence of overlapping grids

D2P9 Model: Numeric Test - L^∞ Error of U_x versus ds



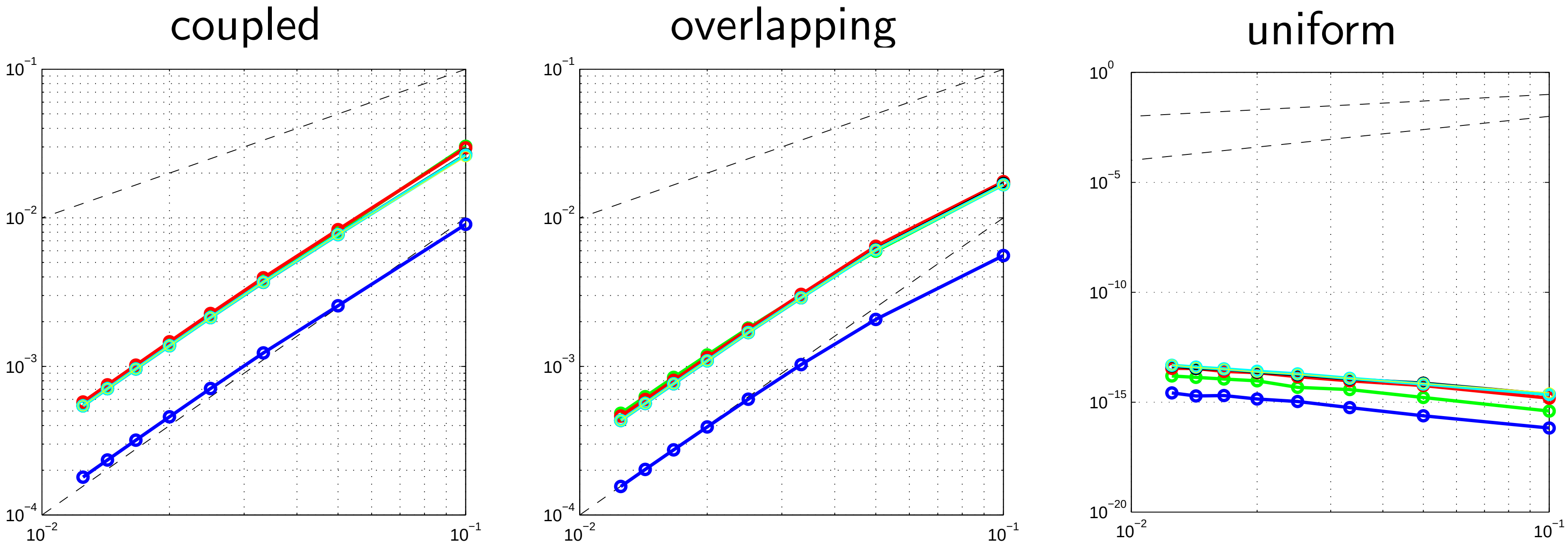
Convergence rates:

(Linear least square fit)

	time	coupled	overlapping	uniform
–	0.1	1.782	1.837	1.870
–	0.2	1.730	1.726	2.118
–	0.4	1.909	1.905	1.983
–	0.6	1.897	1.860	1.980
–	0.8	1.866	1.827	1.979
–	1.0	1.871	1.808	1.979

Setting: nu=0.01, M=10, SWx=3, SWy=3, NEx=8, NEy=8, N=2, maxIt=150, Step=5, InitMode=1, InitPhase=0.5,Frequency=1, smoothStart with forcing, Method=v5cubic(Coupling)/spline(Overlapping)

D2P9 Model: Numeric Test - L^∞ Error of P versus ds



Convergence rates:

(Linear least square fit)

	time	coupled	overlapping	uniform
–	0.1	1.885	1.743	-1.782
–	0.2	1.917	1.718	-1.781
–	0.4	1.896	1.776	-1.512
–	0.6	1.875	1.790	-1.404
–	0.8	1.872	1.781	-1.462
–	1.0	1.862	1.780	-1.375

Setting: nu=0.01, M=10, SWx=3, SWy=3, NEx=8, NEy=8, N=2, maxIt=150, Step=5, InitMode=1, InitPhase=0.5,Frequency=1, smoothStart with forcing, Method=v5cubic(Coupling)/spline(Overlapping)