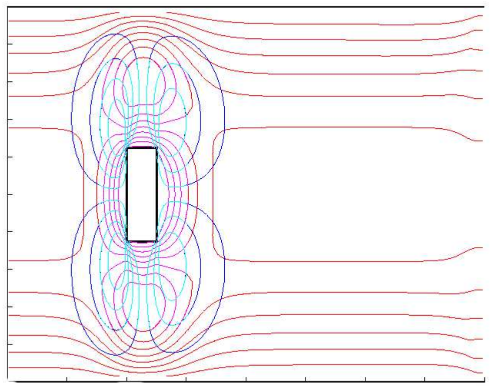
A Consistent Grid Coupling Method for Lattice-Boltzmann Schemes



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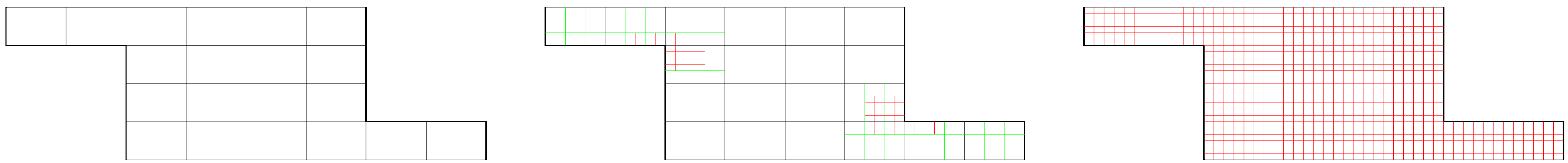
Overview

- Introduction to LB schemes and their analysis
- Paradigmatic model: D1P3 (advection-diffusion)
- Coupling condition & Coupling algorithm
- Numerical tests (convergence study, error snapshots)
- Grid coupling for the D2P9 model (Stokes flow)

Why grid coupling ?

fast, inaccurate $\overset{\text{compromise}}{\longleftrightarrow}$ slow, precise



Lattice-Boltzmann Schemes

TARGET PROBLEM
IBVP for PDE
macroscopic quantities

→ ←

LB ALGORITHM
mesoscopic primary variables
populations: $\mathsf{F}_{\mathbf{s}}(n, \mathbf{i})$

- Velocity space: $\mathcal{S} \subset \{-1, 0, 1\}^d$ (DdPb model on cubic grid $b := \#\mathcal{S}$)

- Moments: $M := \sum_{\mathbf{s} \in \mathcal{S}} \mu(\mathbf{s}) \mathsf{F}_{\mathbf{s}}$ $\longrightarrow$ macroscopic quantities

- LBGK equation on a cubic space grid with mesh size h:

$$\mathsf{F}_{\mathbf{s}}(n+1, \mathbf{i}+\mathbf{s}) = (1-\omega)\mathsf{F}_{\mathbf{s}}(n, \mathbf{i}) + \omega \mathsf{E}_{\mathbf{s}}(n, \mathbf{i}) + h^2 \mathsf{Q}_{\mathbf{s}}(n, \mathbf{i})$$

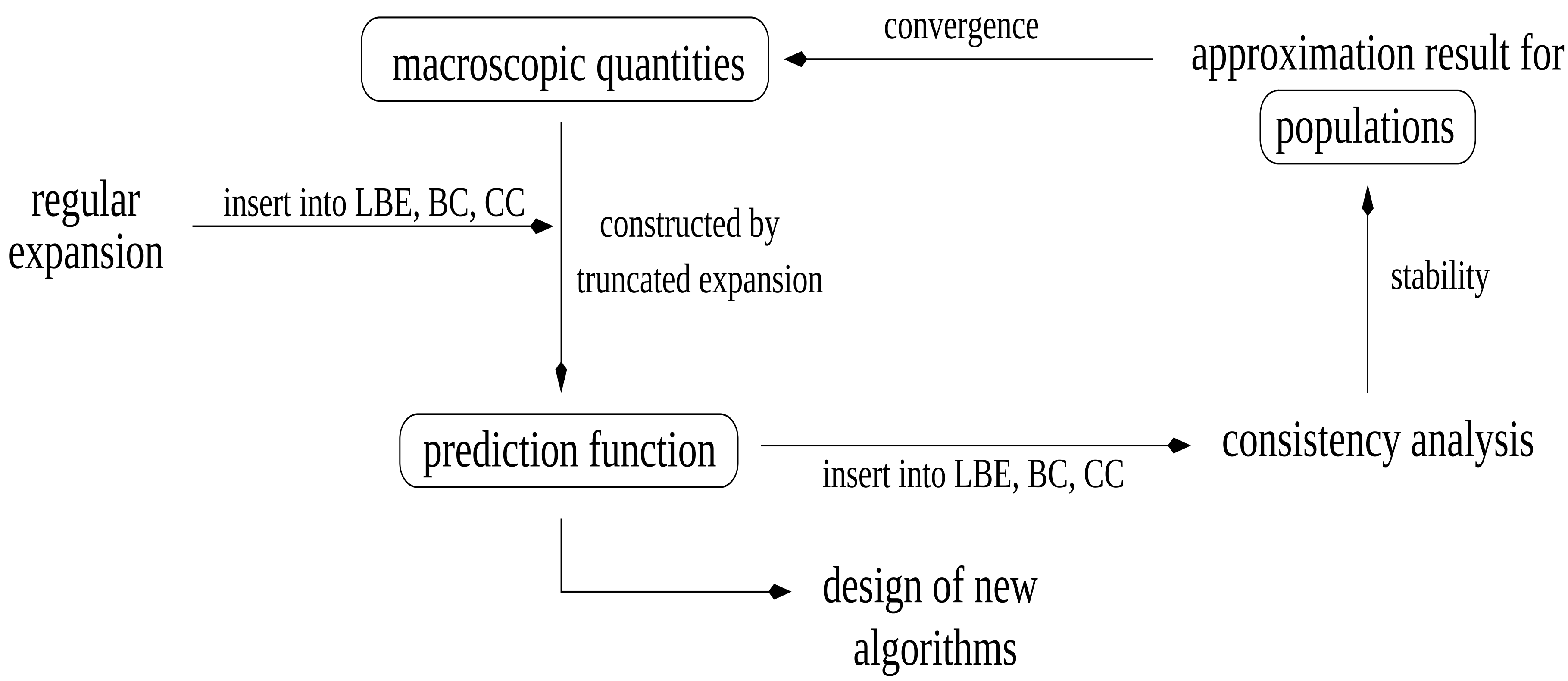
- Equilibrium $\mathsf{E}_{\mathbf{s}} = \mathcal{E}_{\mathbf{s}}\big(M_0(n, \mathbf{i}), \ldots\big)$

- Indices: $t_n = nh^2$, $x_{\mathbf{i}} = \mathbf{i}h$ (diffusive scaling)

Analysis of LB Algorithms

Regular expansion w.r.t. to mesh size h (*Junk et al.*):

$$\mathsf{F}_s(n, \mathbf{i}) \;=\; \mathsf{f}_s^{(0)}(nh^2, \mathbf{i}h) \;+\; h\,\mathsf{f}_s^{(1)}(nh^2, \mathbf{i}h) \;+\; h^2\,\mathsf{f}_s^{(2)}(nh^2, \mathbf{i}h) \;+\; \ldots$$



An Exemplary LB Model: D1P3

$$\left.\begin{array}{r} u(0,\cdot) = u_0 \\ \partial_t u \;+\; a\partial_x u \;-\; \nu\,\partial_x^2 u = q \end{array}\right\} \quad \leftrightarrow \quad \left\{\begin{array}{rcl} \mathcal{S} & = & \{-1,0,1\} \equiv \{\ominus, 0, \oplus\} \\ \mathcal{E}_s(U) & = & \mathsf{w}_s U \;+\; h\,\theta\,s\,\mathsf{w}_s\,a\,U \\ \nu & = & \frac{1}{\theta}(\frac{1}{\omega} - \frac{1}{2}) \end{array}\right.$$

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- Macroscopic quantities: $u(t,x)$, $f(t,x) := \left[a - (\nu + \frac{1}{2\theta})\partial_x\right] u(t,x)$
- 0^{th} & 1^{st} asymptotic order: $\mathsf{f}_s^{(0)} := \mathsf{w}_s u$, $\mathsf{f}_s^{(1)} := \theta \mathsf{w}_s s f$

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- Prediction function: $\mathsf{f}_s(t,x) := \mathsf{f}_s^{(0)}(t,x) + h\mathsf{f}_s^{(1)}(t,x)$
- Approximation result: $\mathsf{F}_s(n,i) = \mathsf{f}_s(nh^2, ih) + \mathrm{O}(h^2)$

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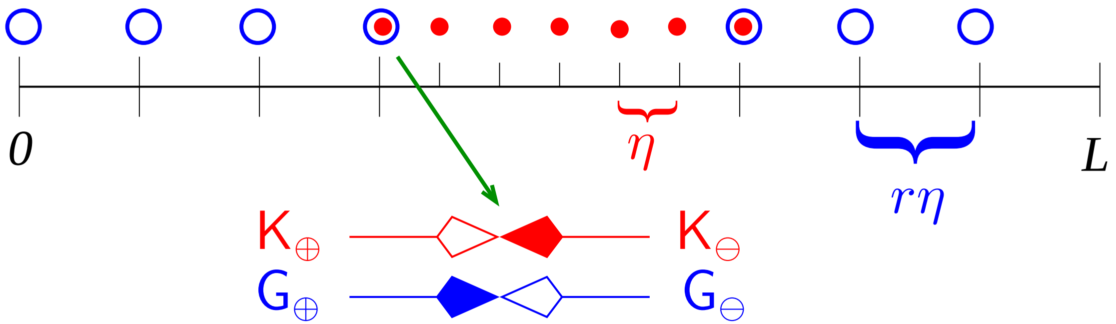
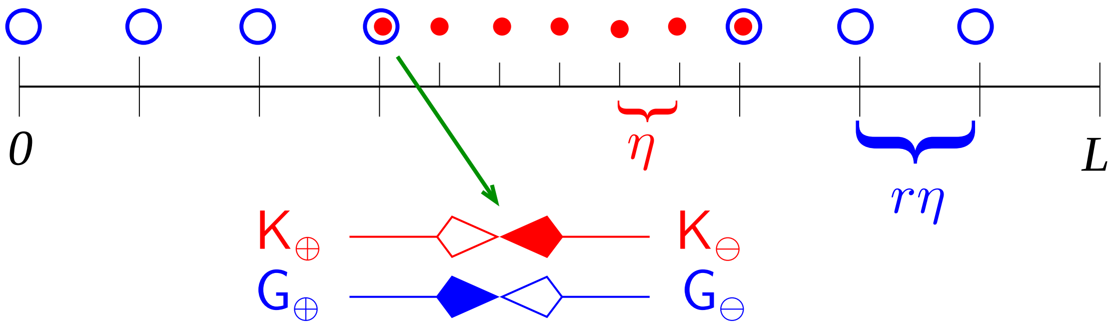
$$\Longrightarrow \left\{\begin{aligned} U(n,i) &:= \sum_{s\in\mathcal{S}} \mathsf{F}_s(n,i) = u(nh^2, ih) + \mathrm{O}(h^2) && 0^{\text{th}} \text{ moment} \\ F(n,i) &:= h^{-1}\sum_{s\in\mathcal{S}} s\,\mathsf{F}_s(n,i) = f(nh^2, ih) + \mathrm{O}(h^2) && 1^{\text{st}} \text{ moment} \end{aligned}\right.$$

The Coupling Condition



	known	empty
coarse:	$G_\oplus$, G_0	$G_\ominus$
fine :	$K_\ominus$, K_0	$K_\oplus$

The Coupling Condition



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Direct exchange of populations ? $\longrightarrow$ vary with h $\Rightarrow$ ☹: 1^{st} order CC

Grid transformation of populations $\longrightarrow$ overlap of subgrids

i) equilibrium/non-equilibrium part: *(Filippova, Hänel, Krafczyk, Chopard)*

ii) preservation of moments (not yet published)

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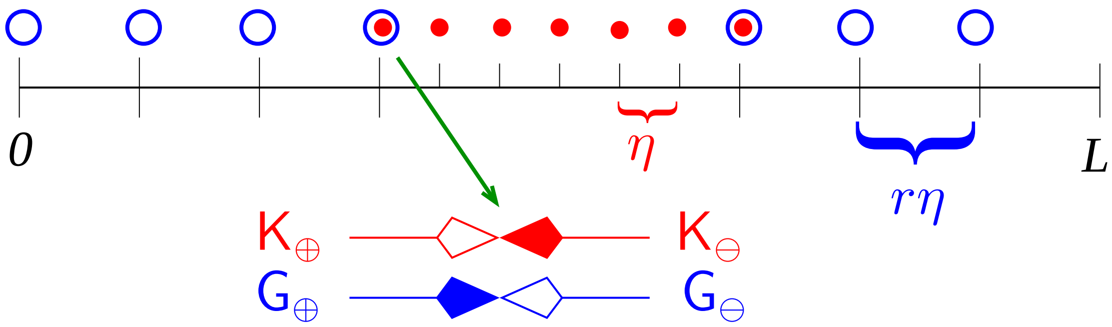

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i) equilibrium/non-equilibrium part: *(Filippova, Hänel, Krafczyk, Chopard)*

ii) preservation of moments (not yet published)

> Idea: equate quantities varying with h^2 (moments) $\leftarrow$ continuity of u, f

implicit cond.: $\sum_{s \in \mathcal{S}} G_s = \sum_{s \in \mathcal{S}} K_s$, $\quad \frac{1}{r\eta} \sum_{s \in \mathcal{S}} s\, G_s = \frac{1}{\eta} \sum_{s \in \mathcal{S}} s\, K_s$

explicit cond.:

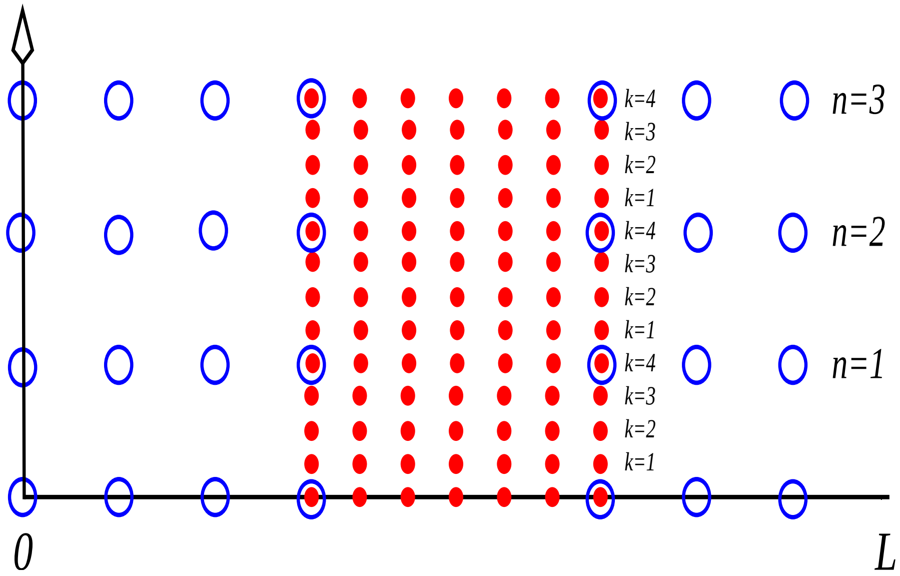
$$\begin{aligned} G_\ominus - K_\oplus &= K_0 + K_\ominus - G_0 - G_\oplus \\ G_\ominus + rK_\oplus &= rK_\ominus + G_\oplus \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 1 & r \end{pmatrix}$$

The Coupling Algorithm

refinement factor $r \in \mathbb{N}$

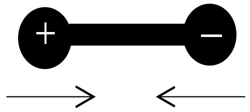
fine grid-size:	η	$\Rightarrow$	coarse grid-size:	$r\eta$	$\Rightarrow$
fine time-step:	η^2	$\Rightarrow$	coarse time-step:	$r^2\eta^2$	

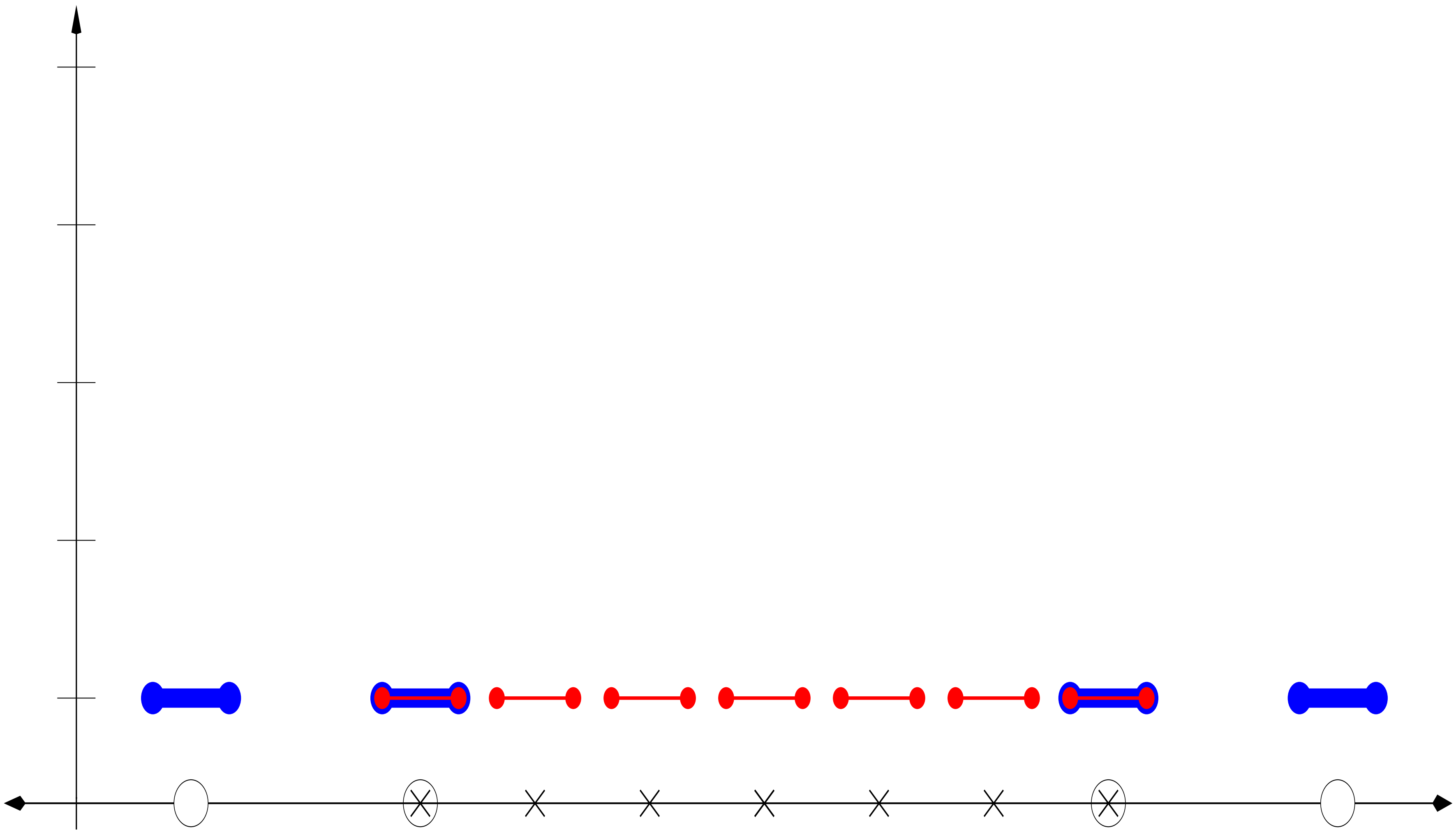
linear interpolation in time at coarse grid interface for intermediate time steps



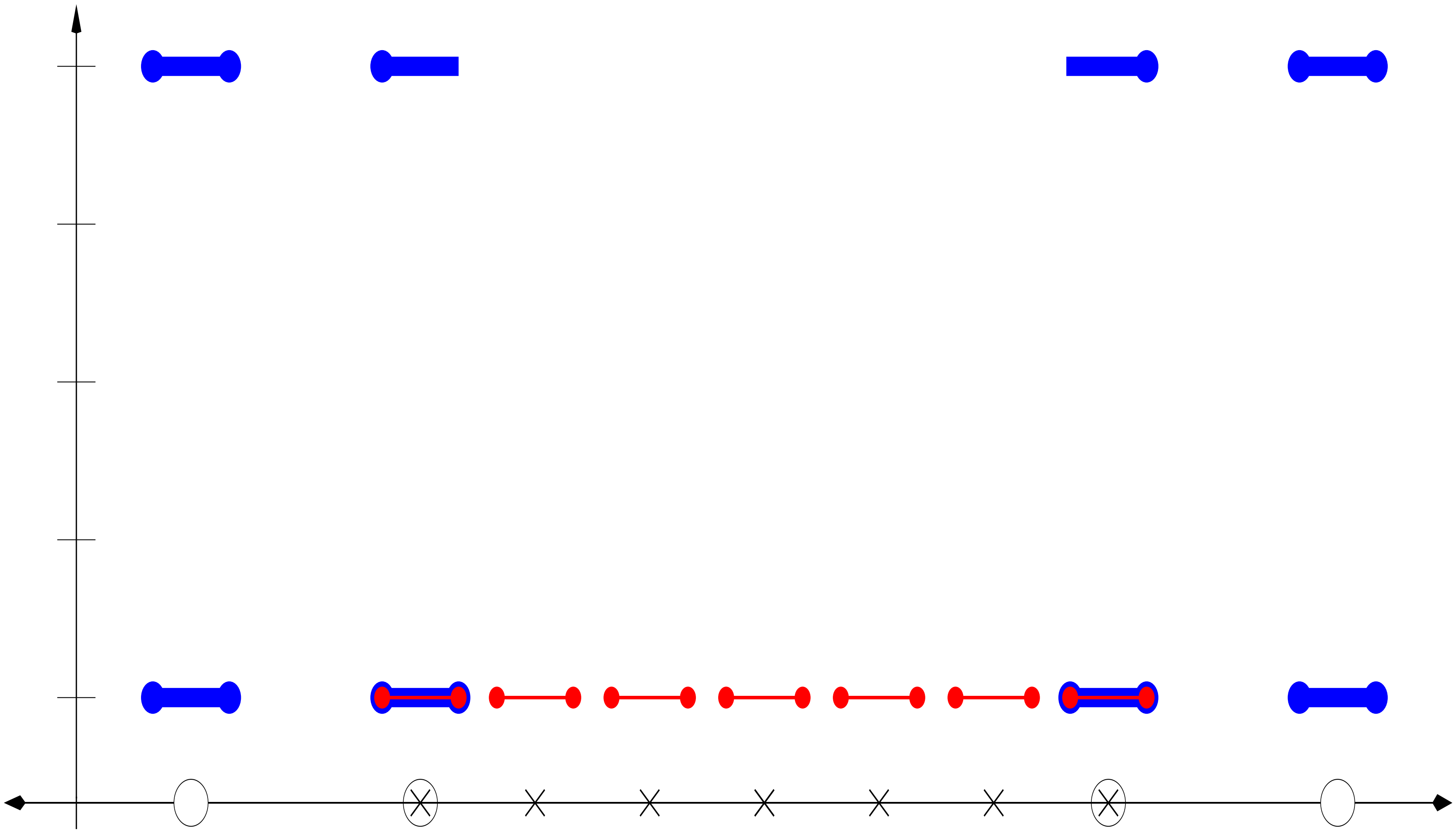
global TimeStep:
collide & propagate on coarse-grid
interpolate known coarse-grid interface-pops
repeat r^2 times
  collide & propagate on fine-grid
  fill empty fine-grid interface-pops
end
fill empty coarse-grid interface-pops

00/11) start: all populations known

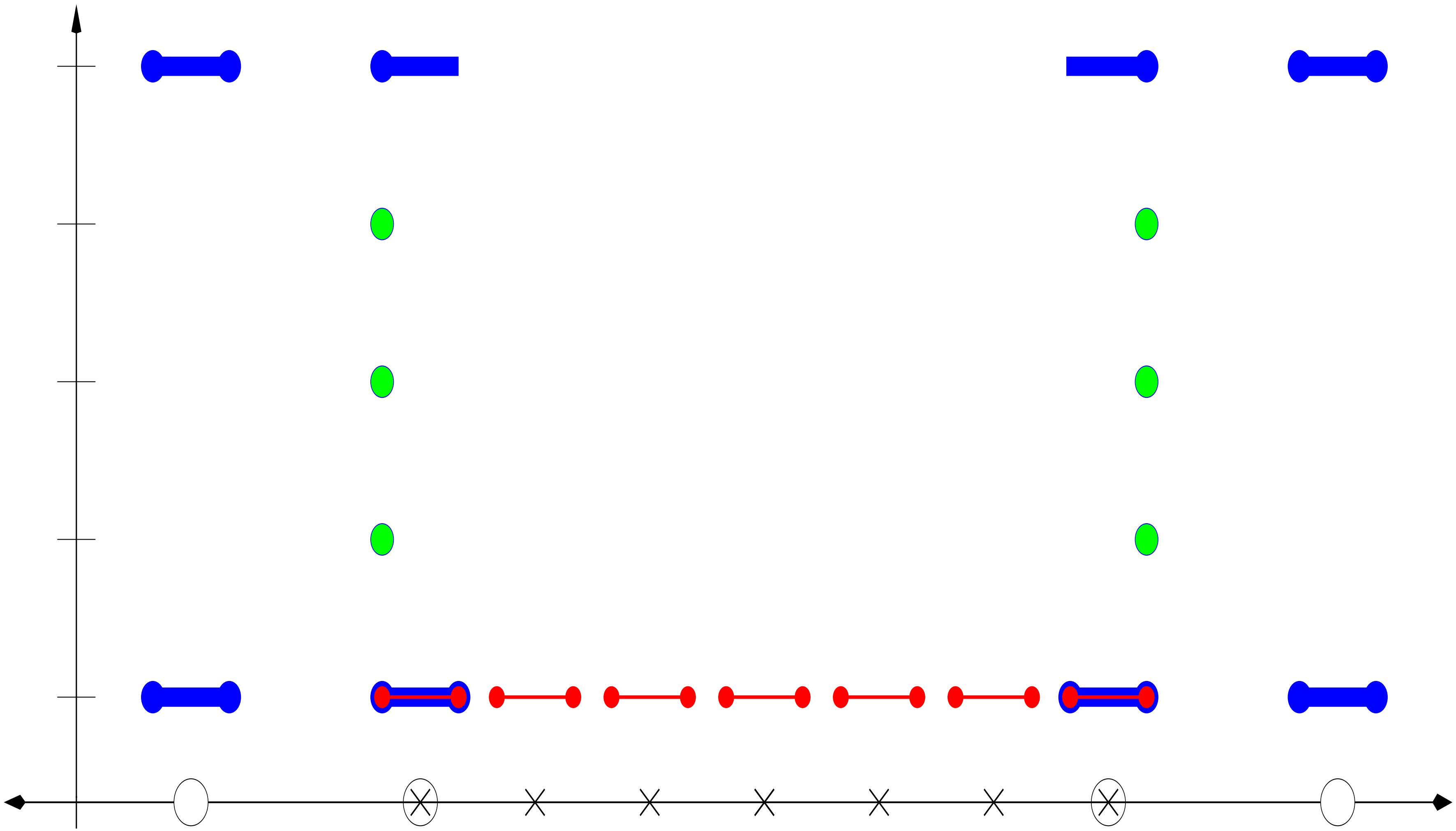




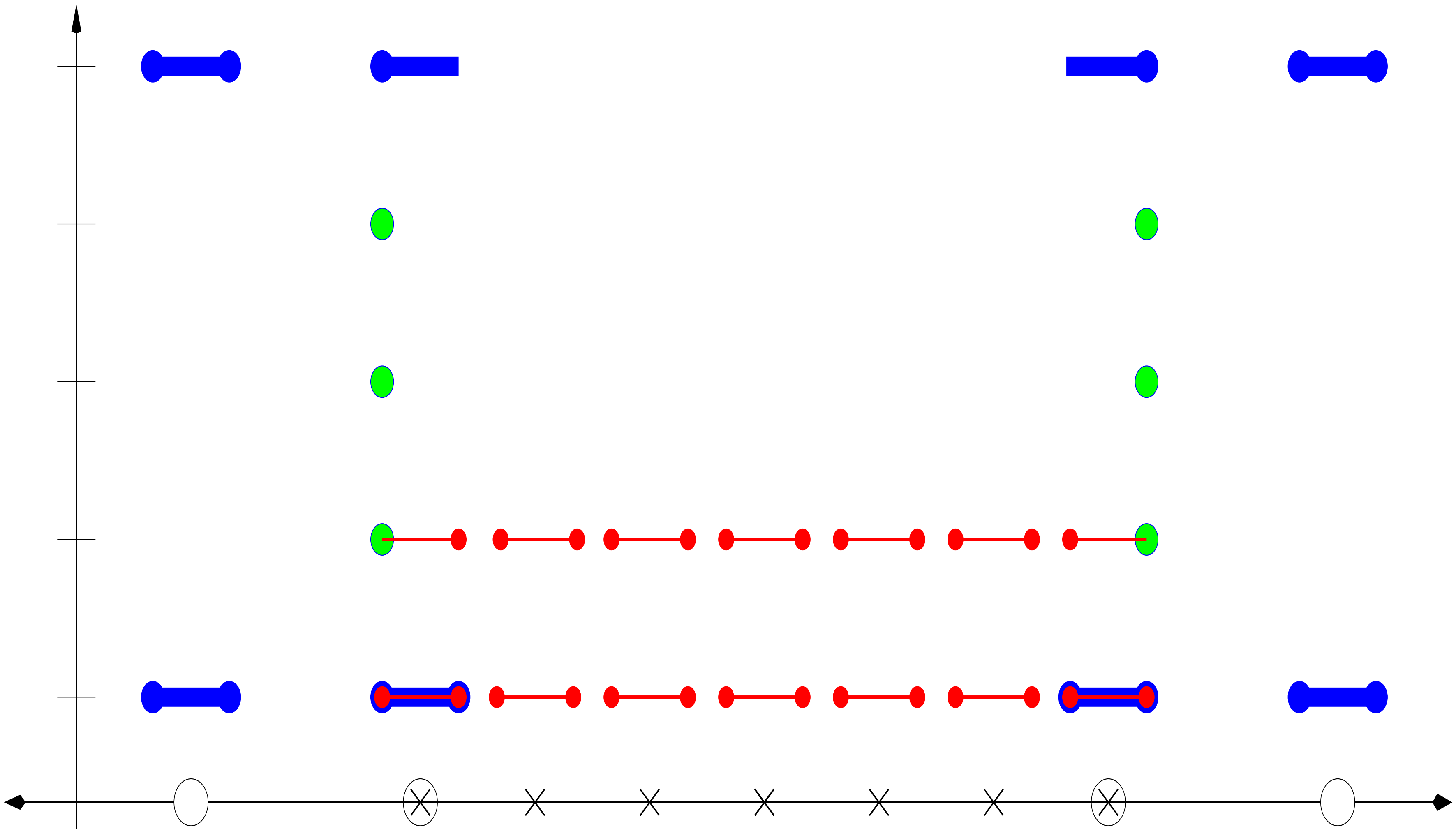
01/11) collide & propagate on coarse-grid



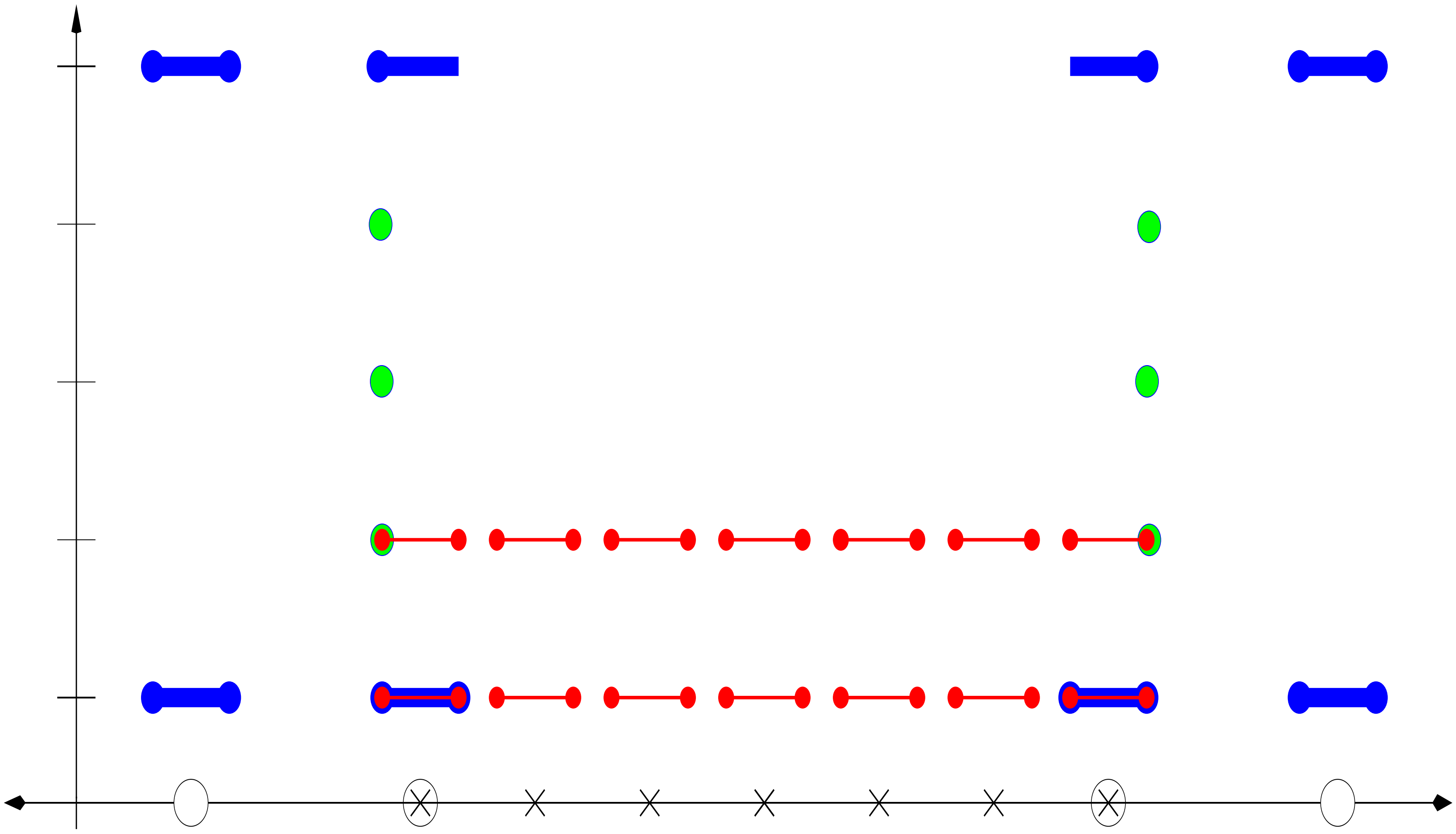
02/11) interpolate known coarse-grid interface-pops



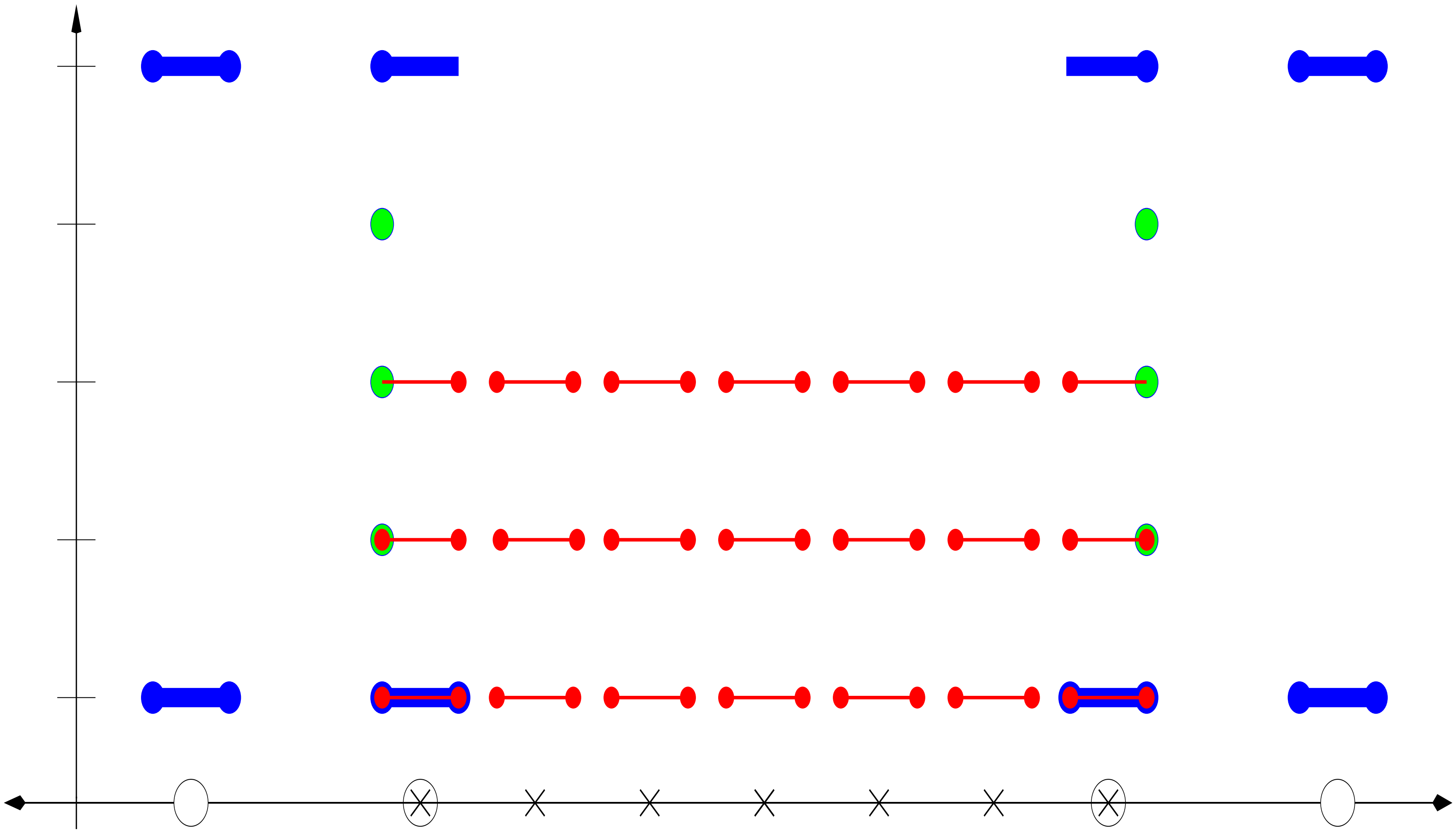
03/11) **1: collide & propagate on fine-grid**



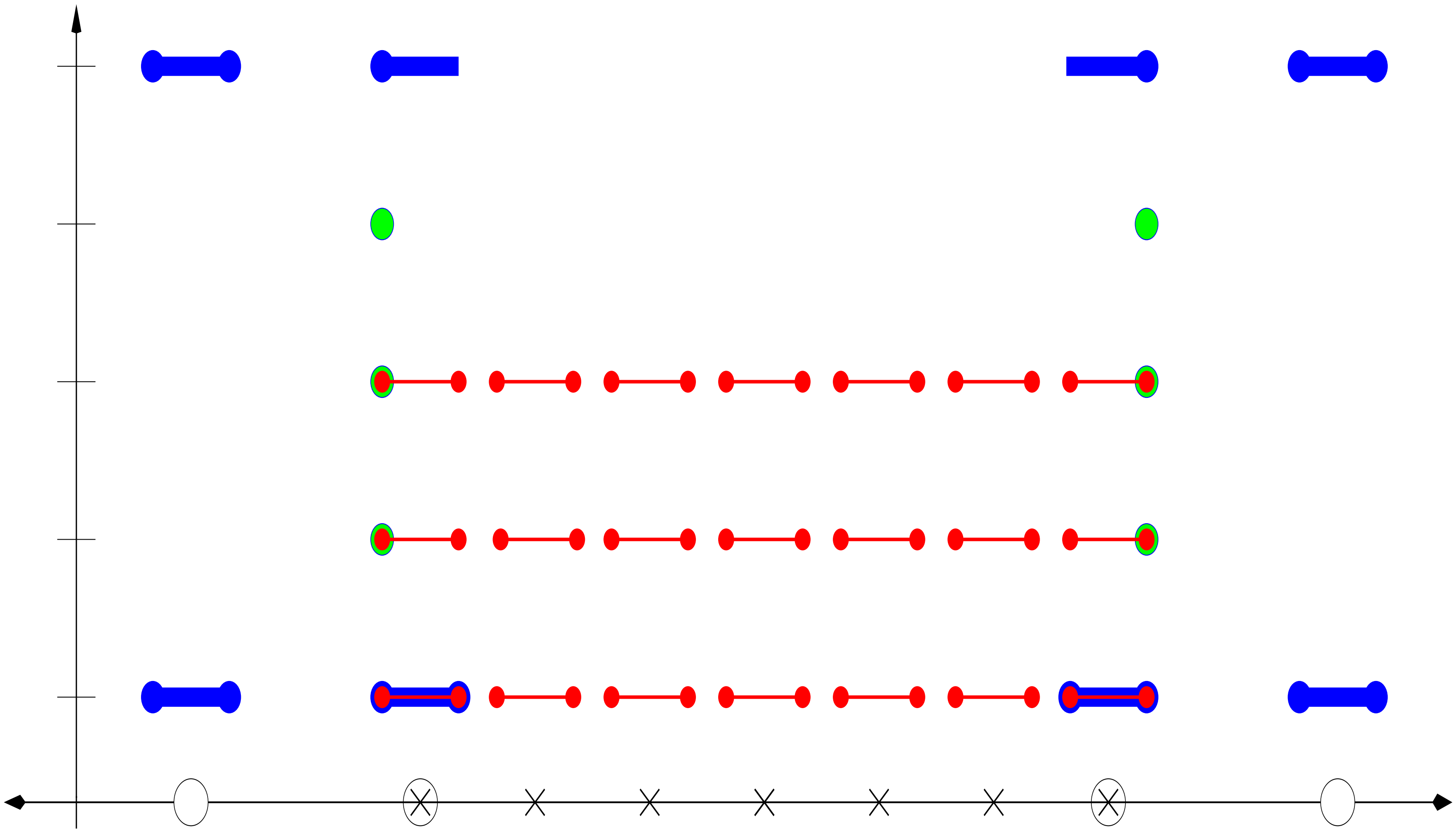
04/11) 1: fill empty fine-grid interface-pops



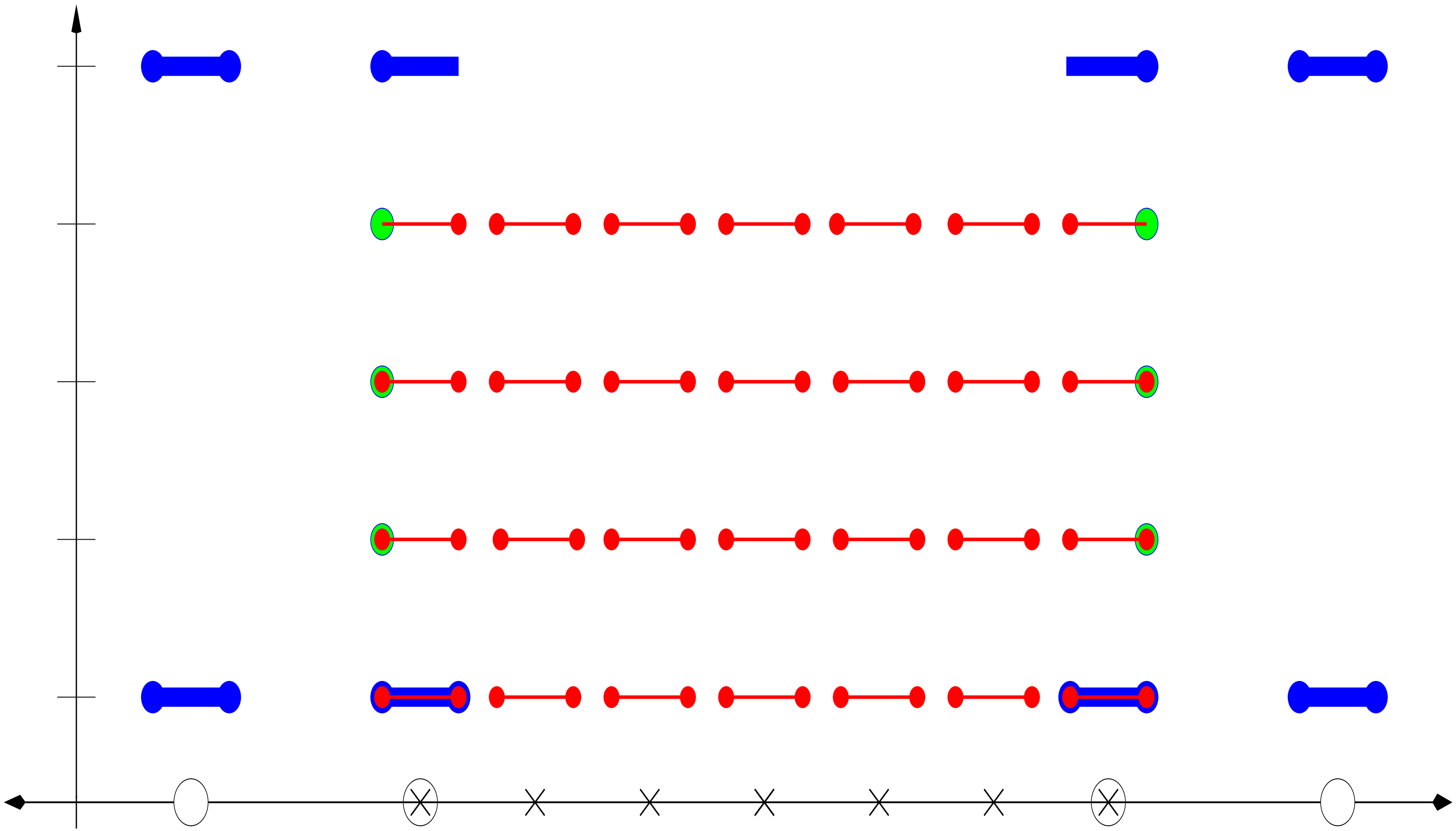
05/11) 2: collide & propagate on fine-grid



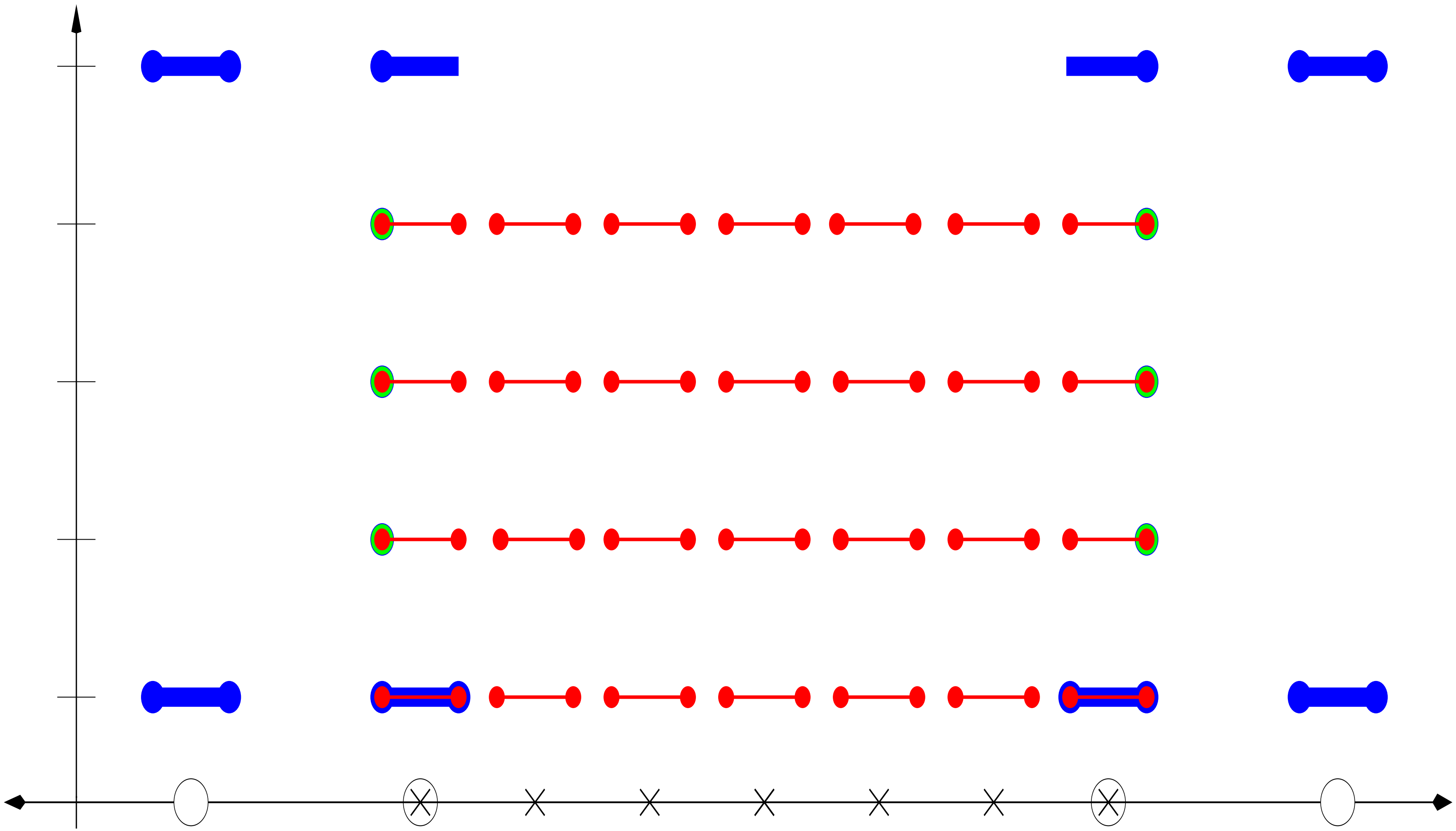
08/11) 2: fill empty fine-grid interface-pops



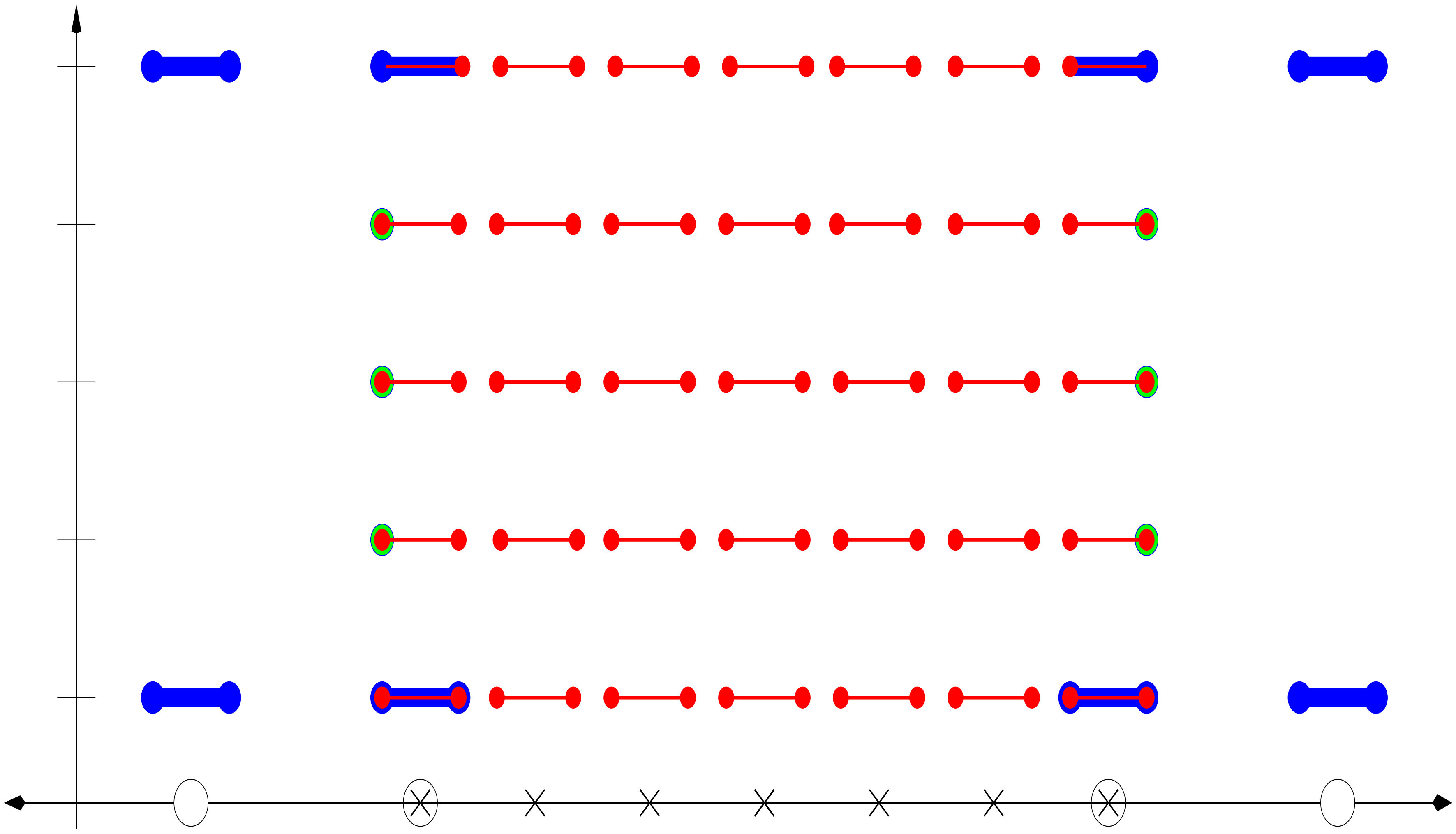
07/11) 3: collide & propagate on fine-grid



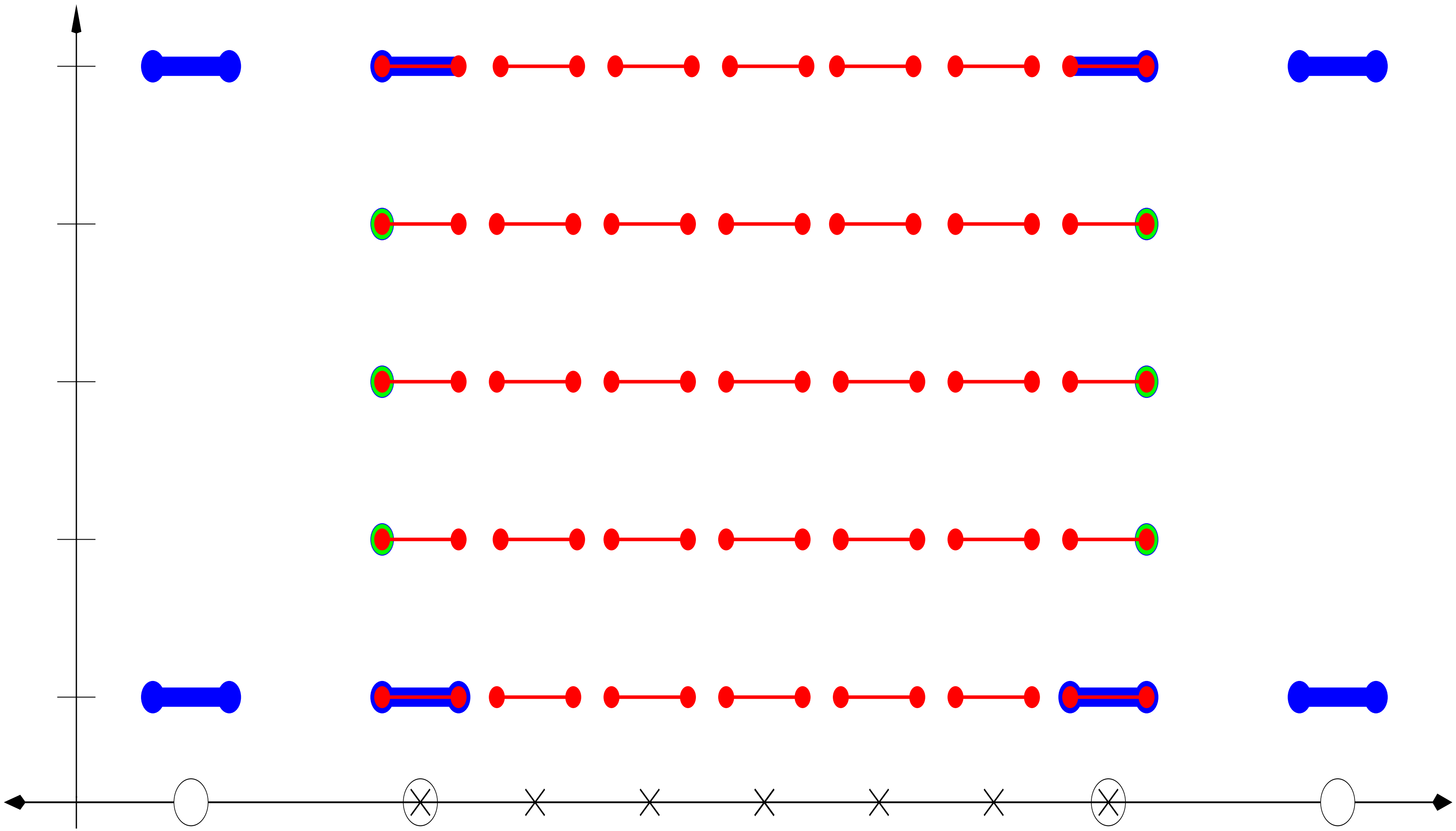
3: fill empty fine-grid interface-pops



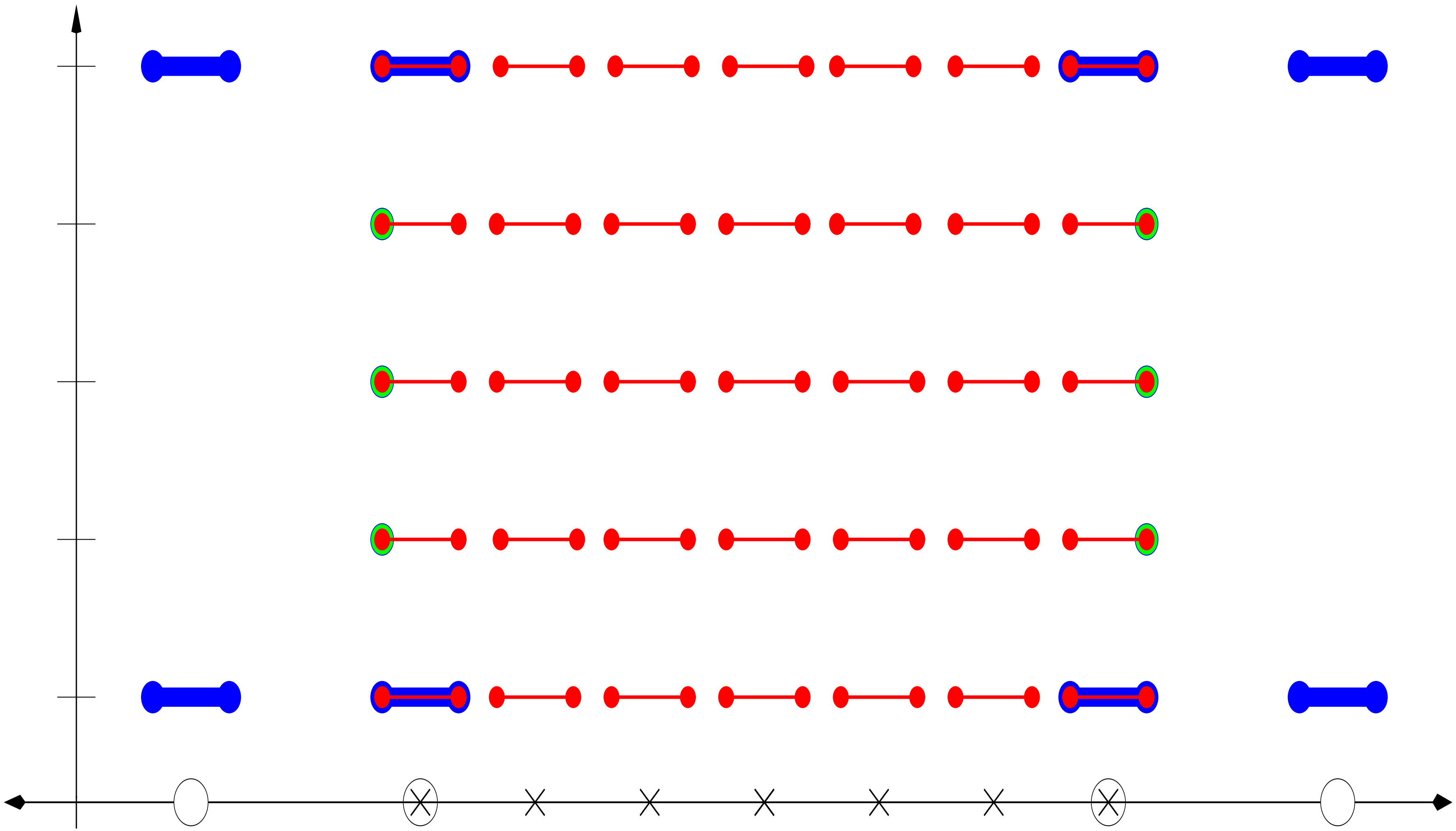
09/11) 4: collide & propagate on fine-grid



10/11) 4: fill empty fine-grid interface-pops



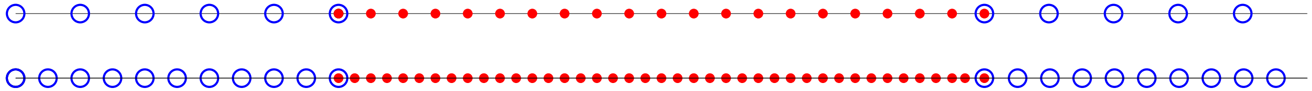
11/11) fill empty coarse-grid interface-pops



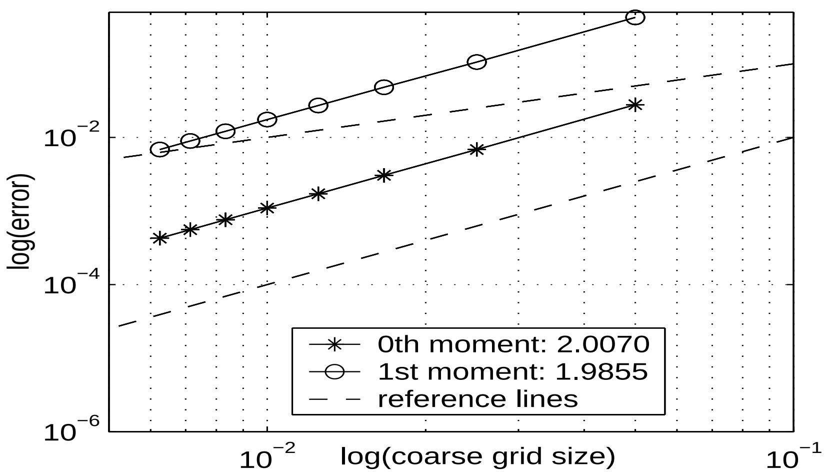
D1P3 Model: Numerical Tests

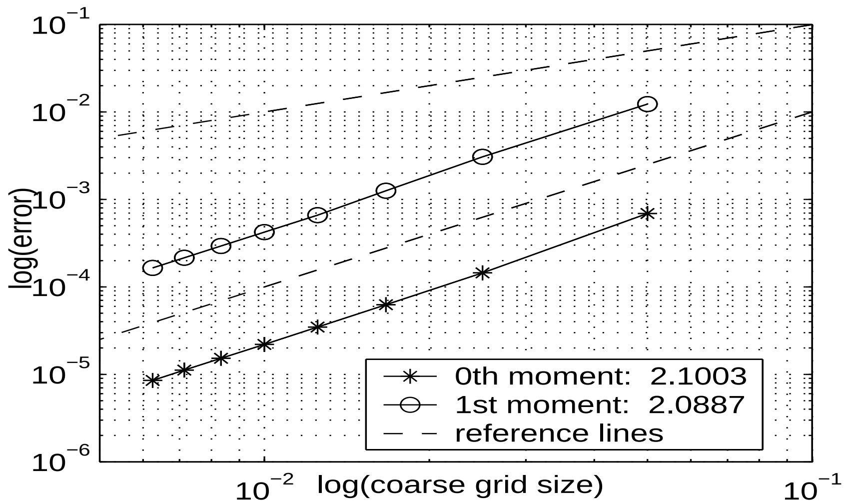
$$\text{(I)}\ \begin{cases} u(t,x) = e^{-36\pi^2 \nu t}\sin(6\pi x) \\ (t,x) \in [0;2]\times[0;1], \quad \text{same for (II)} \\ \nu = 0.005 \end{cases} \qquad \text{(II)}\ \begin{cases} u(t,x) = \cos(2\pi(x-at)) \\ q(t,x) = 4\pi^2\nu\cos(2\pi(x-at)) \\ \nu = 0.001\,, \quad a = 1.333 \end{cases}$$

$$\text{Init.:} \quad \mathsf{F}_s(0,i) \;=\; \underbrace{\mathsf{w}_s u_0(ih)} + h\,\underbrace{\theta\, s\, \mathsf{w}_s f_0(ih)} + h^2\,\underbrace{\tau\,(1-\theta s^2)\,\mathsf{w}_s g_0(ih)}$$

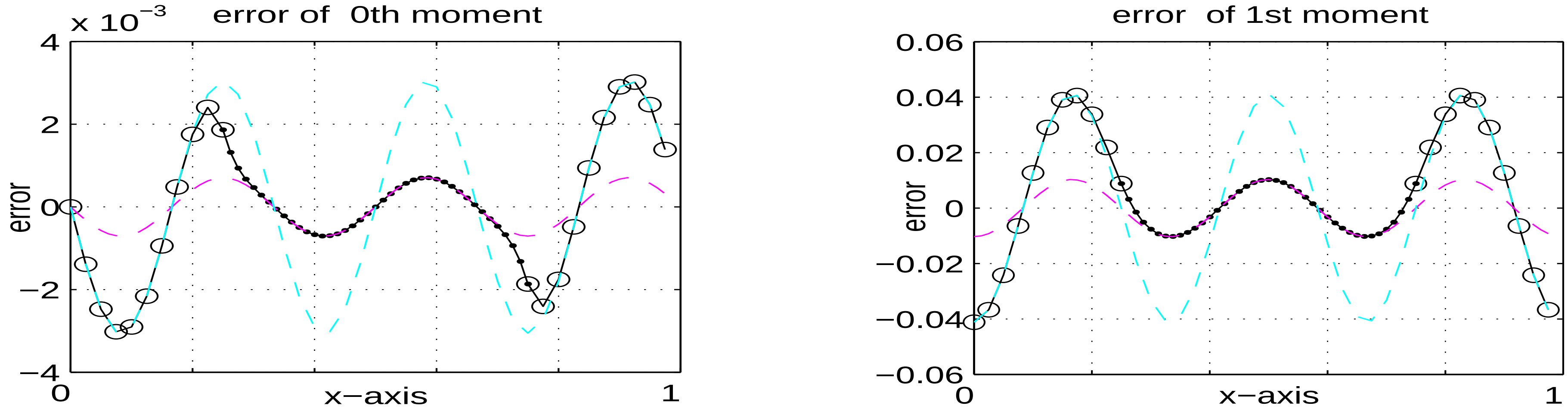


$$g_0 := \partial_x\,(a - \nu\partial_x)\,u_0$$

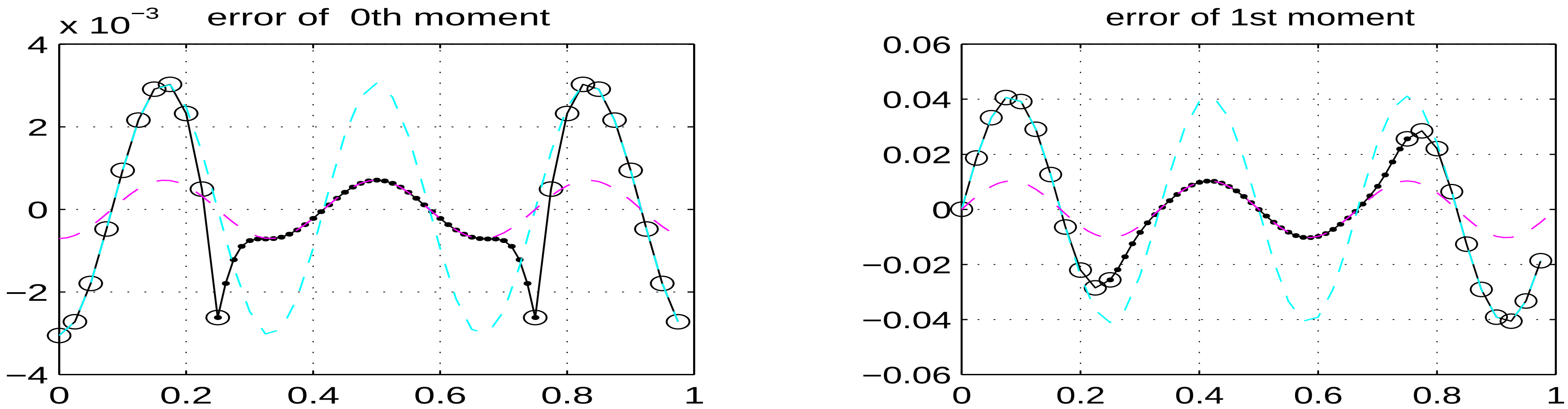




D1P3 Model: Error Snapshots for (I) (t=0.1, 160 It.)



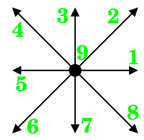
cos instead of sin:



D2P9 Model for Stokes Flow: Macroscopic Quantities

$$\partial_t \mathbf{v} - \nu \Delta \mathbf{v} = -\nabla p \qquad \mathcal{E}_{\mathbf{s}}(p, \mathbf{v}) = 3\,\mathrm{w}_{\mathbf{s}}\,(p + 3\mathbf{s}\cdot\mathbf{v})$$

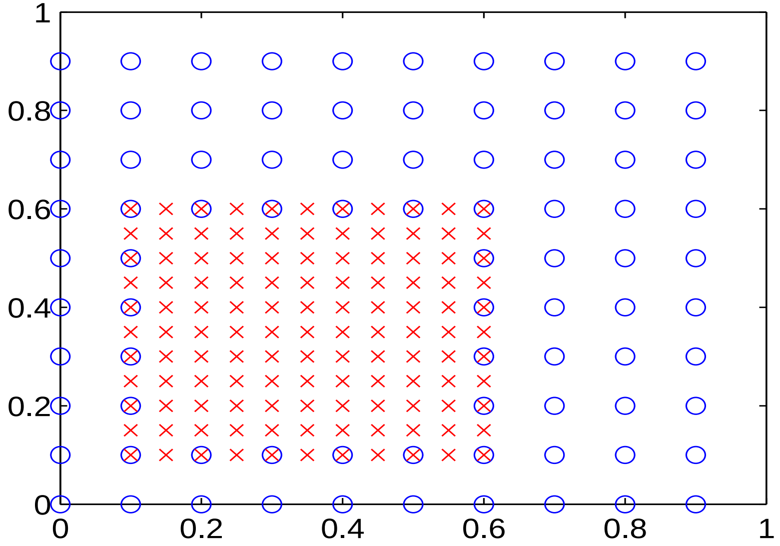
$$\nabla \cdot \mathbf{v} = 0 \qquad \mathbf{v}(0,\cdot) = \mathbf{v}_0 \qquad \nu = \tfrac{1}{3}\left(\tfrac{1}{\omega} - \tfrac{1}{2}\right)$$



p	$\mu_p := \frac{1}{3}h^{-1}$
v_x	$\mu_{v_x} := s_x$
v_y	$\mu_{v_y} := s_y$
$\sigma_x := p - \frac{2}{3}\tau\partial_x v_x$	$\mu_{\sigma_x} := s_x^2 h^{-1}$
$\sigma_y := p - \frac{2}{3}\tau\partial_y v_y$	$\mu_{\sigma_y} := s_y^2 h^{-1}$
$\sigma_{xy} := -\frac{1}{3}\tau(\partial_y v_x + \partial_x v_y)$	$\mu_{\sigma_{xy}} := s_x s_y h^{-1}$
$\phi_x := \frac{2}{9}\tau(\tau - \frac{1}{2})(\partial_x^2 v_y + 2\partial_x\partial_y v_x)$	$\mu_{\phi_x} := (s_x s_y^2 - \frac{1}{3}s_x)h^{-2}$
$\phi_y := \frac{2}{9}\tau(\tau - \frac{1}{2})(\partial_y^2 v_x + 2\partial_x\partial_y v_y)$	$\mu_{\phi_y} := (s_x^2 s_y - \frac{1}{3}s_y)h^{-2}$
$\psi := -\frac{4}{9}\tau(\tau^2 - \tau + \frac{1}{6})\partial_x\partial_y(\partial_x v_y + \partial_y v_x)$	$\mu_\psi := (s_x^2 s_y^2 - \frac{1}{3}(s_x^2 + s_y^2) + \frac{1}{9})h^{-3}$

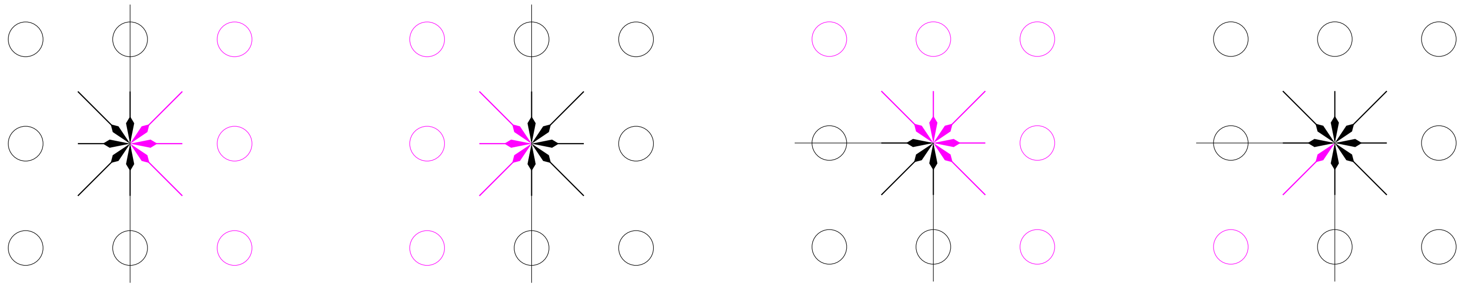
D2P9 Model: Grid Coupling

New features: 1) hanging interface nodes of fine grid, 2) corner nodes



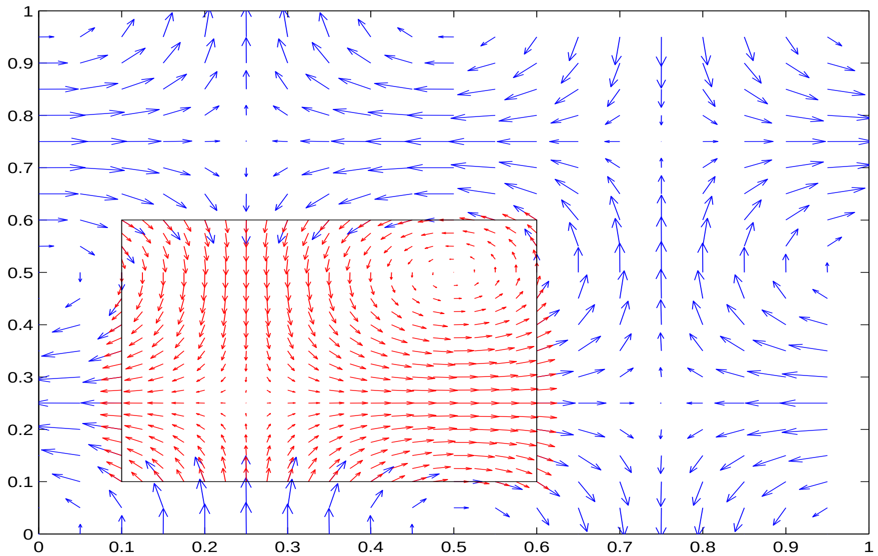
type of interface node	equalized moments
horizontal ($\parallel$ x-axis)	$V_x, V_y, \Sigma_y, \Sigma_{xy}, \Phi_y, \Psi$
vertical ($\parallel$ y-axis)	$V_x, V_y, \Sigma_x, \Sigma_{xy}, \Phi_x, \Psi$
corner	$V_x, V_y, \Sigma_x, \Sigma_y, \Sigma_{xy}, \Psi$

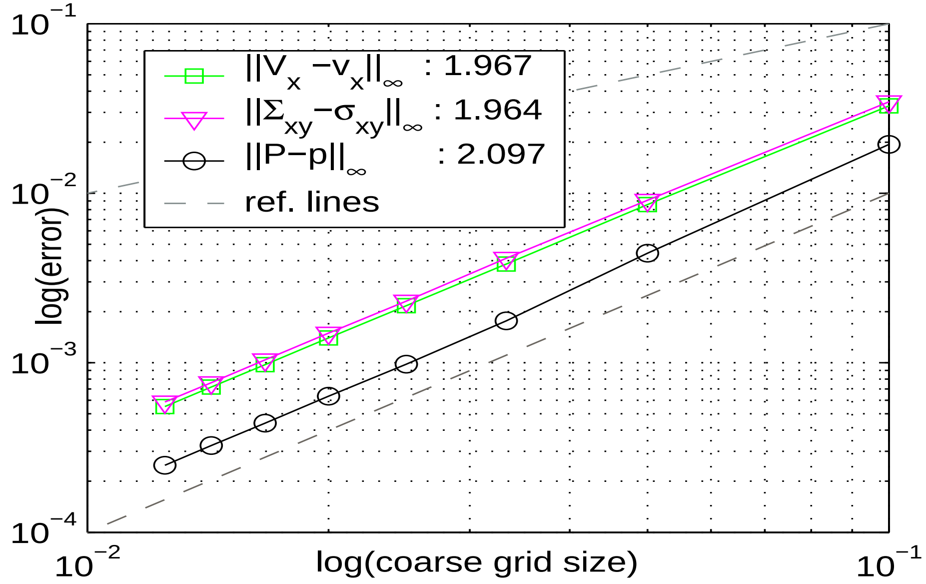
regular interface nodes: convex/concave corner:



6 empty (unknown) populations $\Rightarrow$ 6 conditions (physically only 4 for Stokes !)

D2P9 Model: Numeric Test





$$
\left.
\begin{aligned}
v_x(t,x,y) &= -e^{-8\pi^2\nu t}\cos(2\pi x)\sin(2\pi y) \\
v_y(t,x,y) &= \quad e^{-8\pi^2\nu t}\sin(2\pi x)\cos(2\pi y) \\
p(t,x,y) &= 0
\end{aligned}
\right\}
$$

damped eigenmode of Stokes operator

$\nu = 0.01$, $(t,x,y) \in [0;2] \times [0;1]^2$

Init.: $\mathsf{F}_\mathbf{s}(0,i,j) = \underbrace{3\mathsf{w}_\mathbf{s}\mathbf{s}\cdot\mathbf{v}_0(ih,jh)} + h\,\underbrace{3\mathsf{w}_\mathbf{s}\left[p_0(ih,jh) + \tfrac{1}{\omega}(\mathbf{s}\cdot\nabla)(\mathbf{s}\cdot\mathbf{v}_0(ih,jh))\right]}$