Analysis and Synthesis of Lattice-Boltzmann Methods

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- Boundary Conditions
- Layer Analysis
- Convergence
- Summary

LBM setup and benchmark problem

LBM : $f_i(n+1, \boldsymbol{j}+\boldsymbol{c}_i) = f_i(n, \boldsymbol{j}) - \mathcal{J}_i(f)(n, \boldsymbol{j})$

Moments: $\hat{\rho} = \sum_i f_i, \quad \hat{\boldsymbol{u}} = \sum_i \boldsymbol{c}_i f_i$

Navier-Stokes:

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \nu\Delta\boldsymbol{u},$$

$$\mathrm{div}\boldsymbol{u} = 0, \quad \boldsymbol{u}|_{t=0} = \boldsymbol{\psi}, \quad \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{\varphi}$$

Collison:

$\mathcal{J}(f)(n, \boldsymbol{j}) = A(\boldsymbol{f} - \boldsymbol{f}^{eq})$

A: linear mapping;

$\boldsymbol{f}^{eq}$ equilibrium

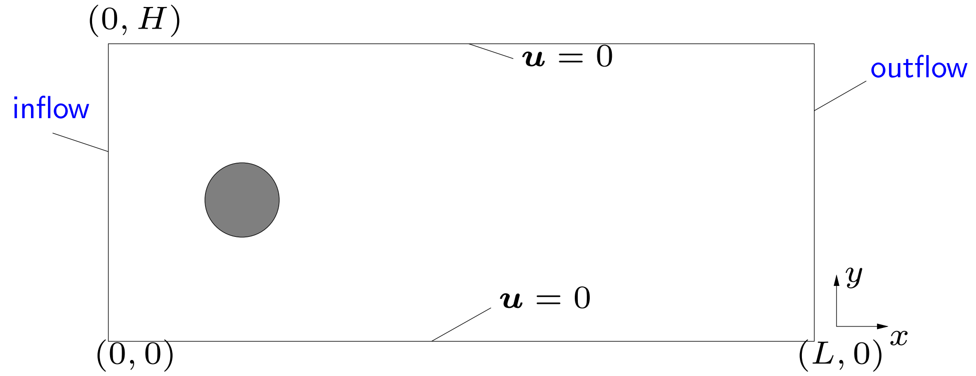
$\boldsymbol{c}_i$: $D2Q9, \quad D3Qx$

Lattice: cubic

Grid size: h

Ideal Consistency: 2^{nd} order

Require: suitable IC, BC

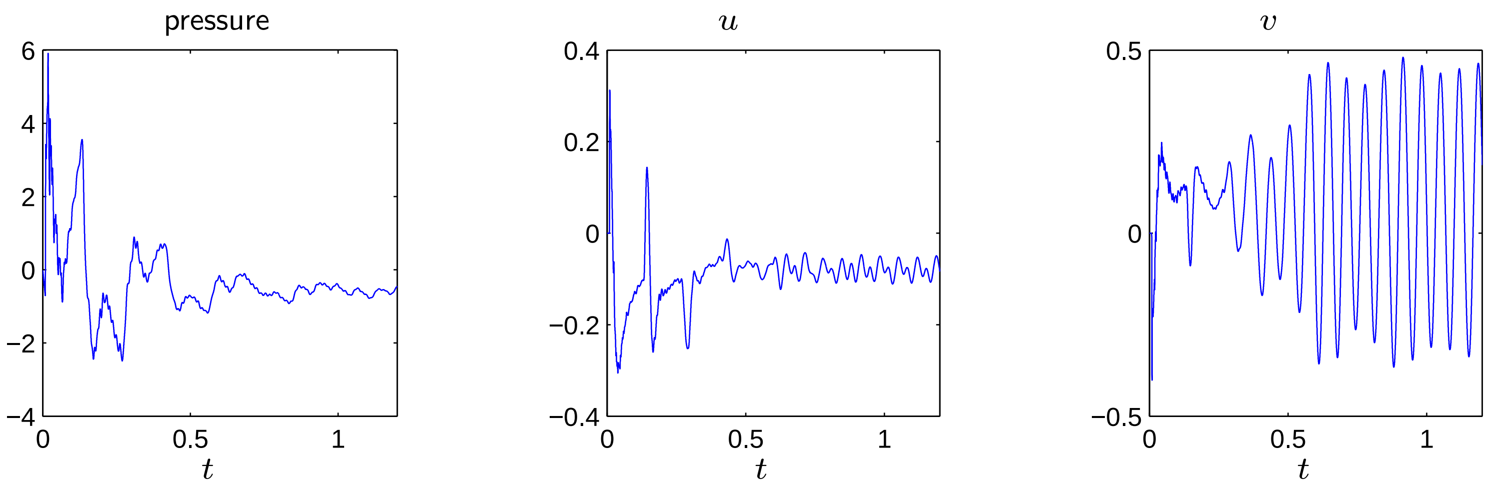


Initial Condition: $f_i(0,\boldsymbol{j}) = f_i^{eq}(\hat{\rho}=1,\hat{\boldsymbol{u}}=0)$

Impulsive inflow: $U_{in} = y(1-y), \qquad V_{in} = 0$

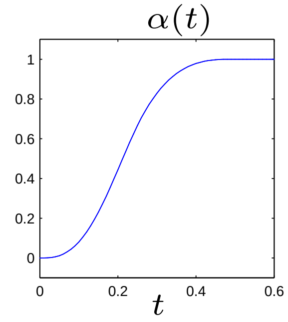
Consistency: 0^{th} order accuracy

Numerical pressure and velocity at a monitor point in the wake:



Initial Condition: $f_i(0, \boldsymbol{j}) = f_i^{eq}(\hat{\rho} = 1, \hat{\boldsymbol{u}} = 0)$

Gradual inflow: $U_{in} = \alpha(t)y(1-y), \qquad V_{in} = 0$



Consistency : 2^{nd} order accuracy

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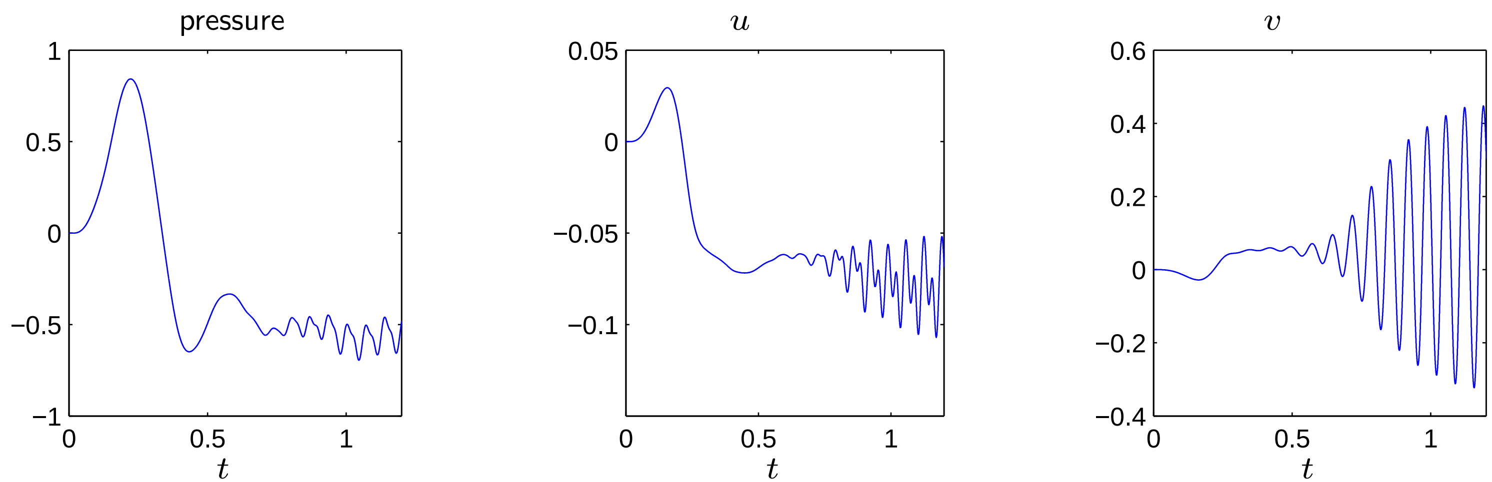
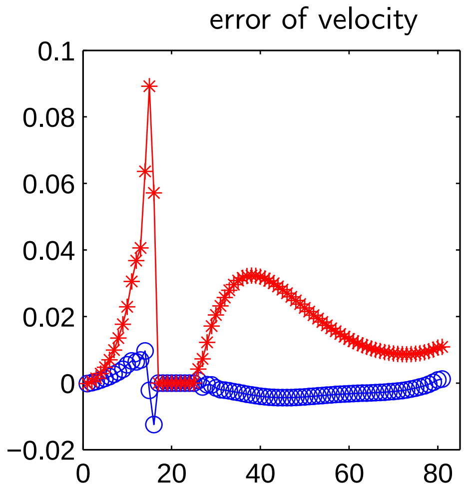
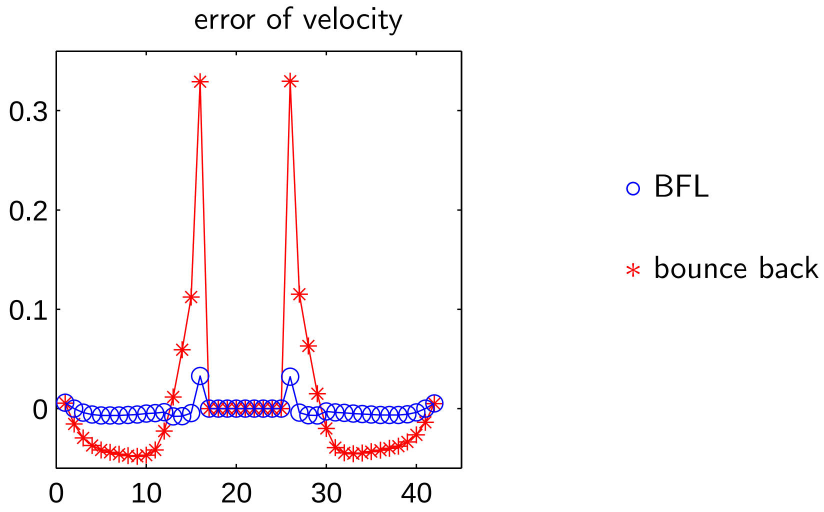


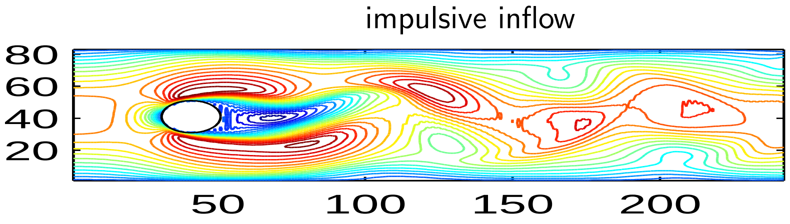
Boundary condition at rigid walls

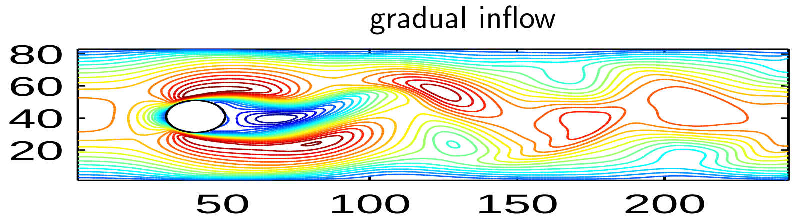
Plane wall: bounce back at $\frac{1}{2}$ link Curved wall: POP_1 or BFL

Consistency: Velocity: 2^{nd} order, Pressure: 1^{st} order









Outflow conditions

- N. Thürey et. al.(TH):

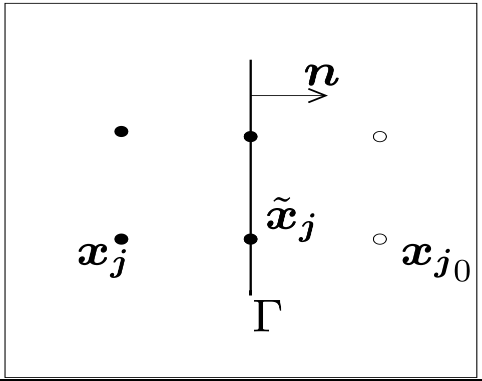
$$f_i(n+1,\boldsymbol{j}) = f_i^{eq}(n,\boldsymbol{j}) + f_{i^*}^{eq}(n,\boldsymbol{j}) - f_{i^*}(n,\boldsymbol{j})$$

Analysis $\Longrightarrow S[\boldsymbol{u}]\boldsymbol{n} = 0$

- Neumann condition(NBC):

$$f_i(n+1,\boldsymbol{j}_0) = f_{i^*}^c(n,\boldsymbol{j}_0 + h\boldsymbol{c}_i) + 2c_s^2 f_i^* \hat{\boldsymbol{u}}(n,\boldsymbol{j}) \cdot \boldsymbol{c}_i$$

Analysis $\Longrightarrow \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}(t,\boldsymbol{x}) = 0$



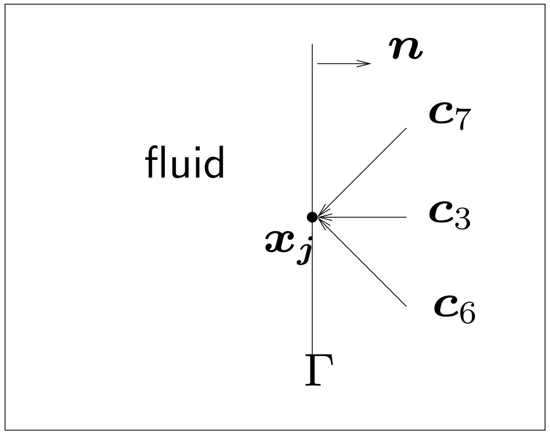
Outflow conditions

- Zero normal Stress condition(ZNS):

$$f_3(n+1,\boldsymbol{j}) = F_3^{eq}(1,\hat{\boldsymbol{u}}(n,\boldsymbol{j})) - ((2\nu A - \boldsymbol{I})(\boldsymbol{f} - \boldsymbol{f}^{eq}))_3,$$

$$f_i(n+1,\boldsymbol{j}) = F_i^{eq}(1,\hat{\boldsymbol{u}}(n,\boldsymbol{j})) - \frac{f_i^*}{f_3^*}(2\nu A(\boldsymbol{f} - \boldsymbol{f}^{eq}))_3\,, \quad i = 6,7$$

Analysis $\Longrightarrow (-p\boldsymbol{I} + 2\nu S[\boldsymbol{u}])\boldsymbol{n} = 0$

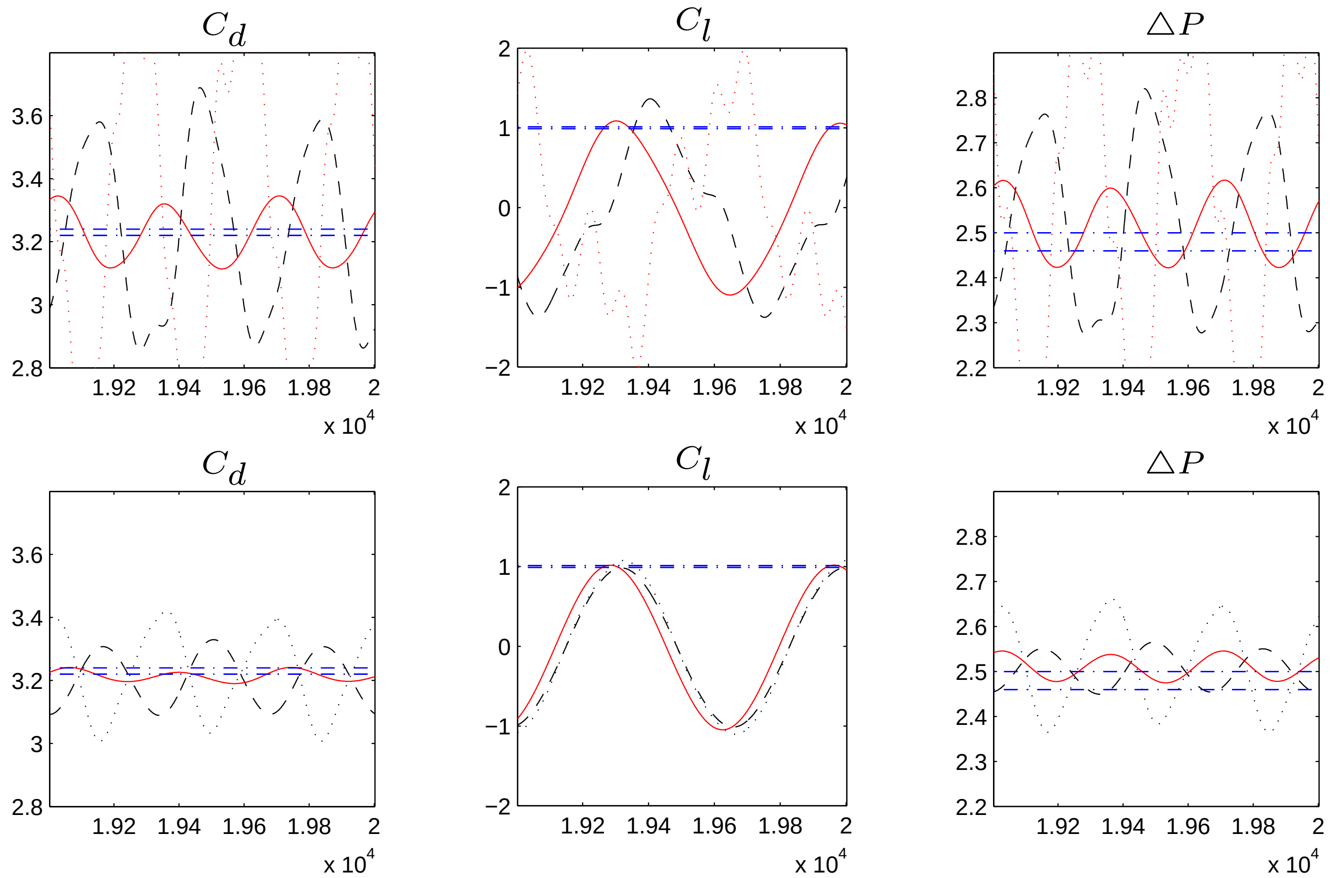


Unsteady flow with $Re = 100$

Comparison of C_d, C_l, $\triangle P$

NBC:

ZNS:

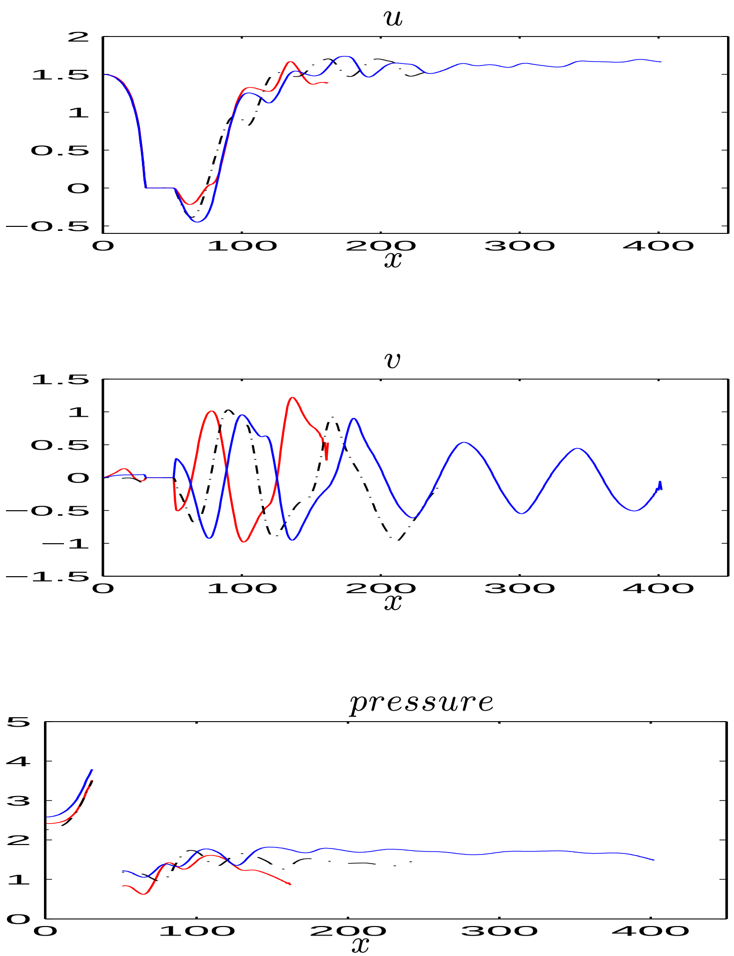


Solid line: $L/H = 5$, dashed line: $L/H = 3$, dotted line: $L/H = 2$.

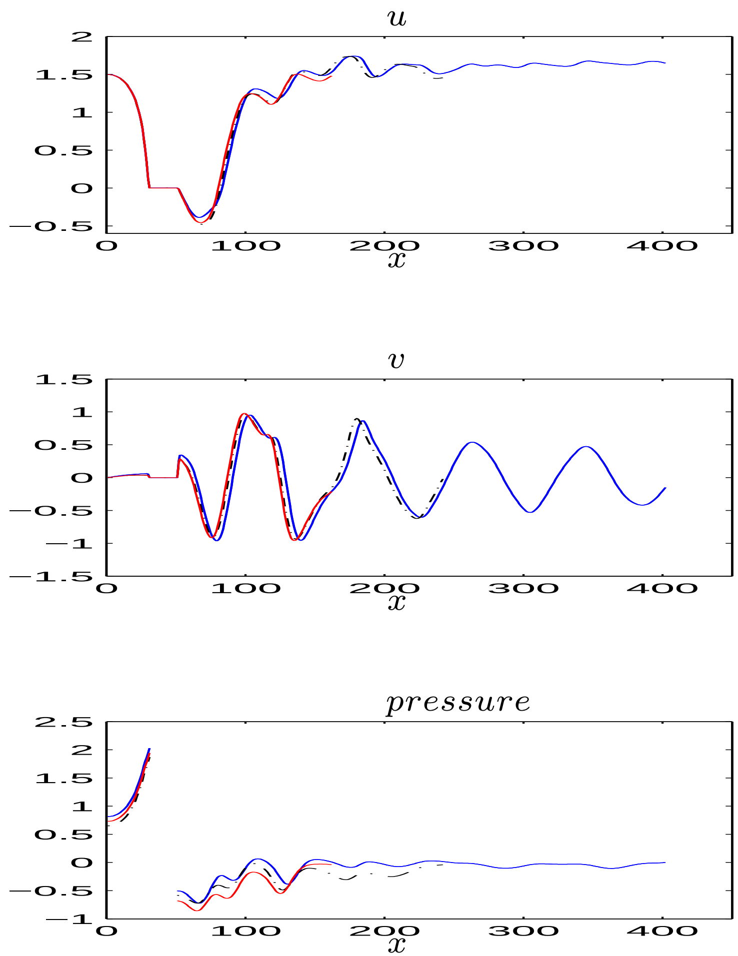
Dash-dotted line: reference values.

Pressure and velocity along the central line

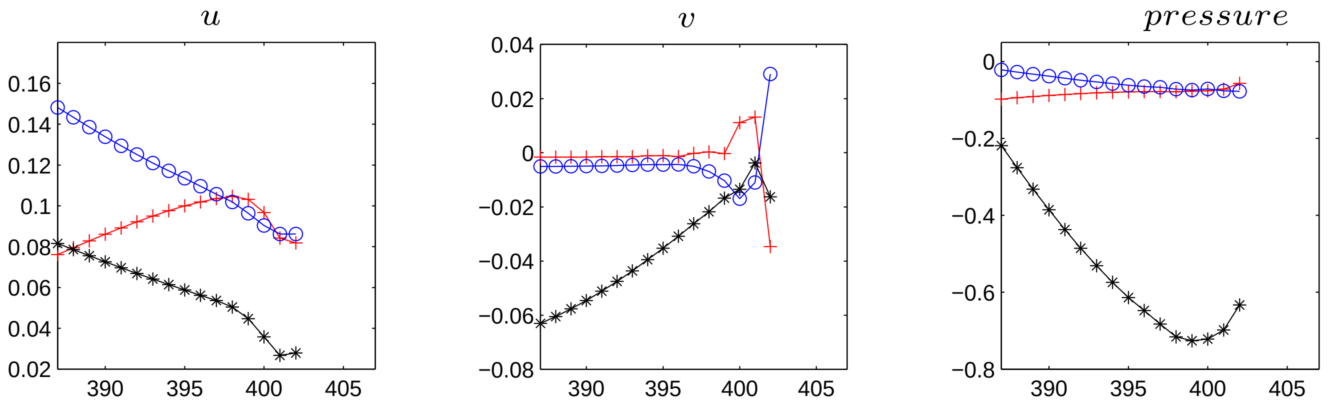
NBC



ZNS



Boundary layers at outflow



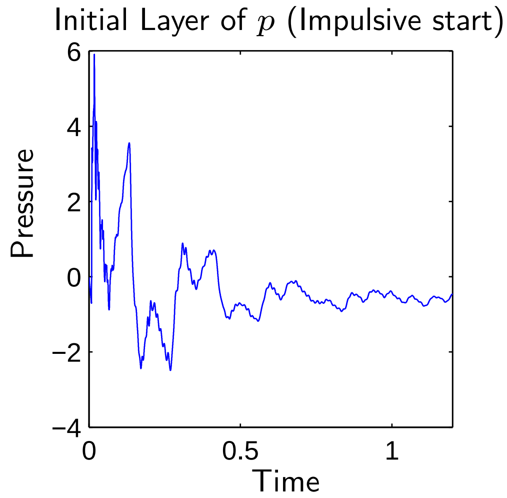
Analysis is necessary!

Overview

- LB-algorithms produce unwanted numerical effects.
- To understand better disturbing phenomena like initial and boundary layers $\longrightarrow$ study model problems.
- Analytically tractable, exhibit similar features despite their simplicity.

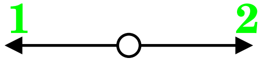
- So far: mainly using regular expansions for the analysis of numerical algorithms/ parameter-depending ODEs/PDEs.
- Here: demonstration by two examples, how to apply irregular expansions.
- Concluding remarks about asymptotic expansions and convergence.

Analysis of an Initial Layer



Aller Anfang ist nicht nur schwer,
sondern verläuft auch selten glatt.

The D1P2 Lattice-Boltzmann Equation (Model Problem)

- Velocity space: 

Population functions: $\mathsf{f}_1, \mathsf{f}_2 : [0, T] \times [0, L] \to \mathbb{R}$

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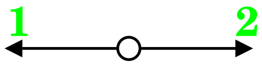
- Velocity space: 

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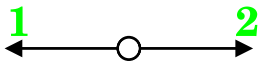
- Velocity space: 

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- LB equation with diffusive scaling:

$$\begin{aligned} \partial_t \mathsf{f}_1 - \epsilon^{-1}\partial_x \mathsf{f}_1 &= -\epsilon^{-2}\omega\big[\mathsf{f}_1 - \tfrac{1}{2}u\big] = -\epsilon^{-2}\tfrac{\omega}{2}\big[\mathsf{f}_1 - \mathsf{f}_2\big] \\ \partial_t \mathsf{f}_2 + \epsilon^{-1}\partial_x \mathsf{f}_2 &= -\epsilon^{-2}\omega\big[\mathsf{f}_2 - \tfrac{1}{2}u\big] = -\epsilon^{-2}\tfrac{\omega}{2}\big[\mathsf{f}_2 - \mathsf{f}_1\big] \end{aligned}$$

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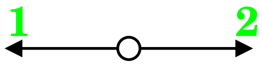
- Velocity space: 

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\end{aligned}
$$

- Linear transformation $\mathsf{f}_1, \mathsf{f}_2 \leftrightarrow u, \phi$ leads to equivalent moment system:

$$
\begin{aligned}
\partial_t u + \partial_x \phi &= 0 & \partial_x\partial_t\phi &= -\partial_t^2 u \\
\partial_t \phi + \epsilon^{-2}\partial_x u &= -\epsilon^{-2}\omega\phi & \partial_x\phi &= -\epsilon^2\tau\partial_x\partial_t\phi - \tau\partial_x^2 u
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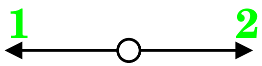
- Velocity space: 

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- Closed scalar equation in u: $\epsilon^2\tau\partial_t^2 u + \partial_t u - \tau\partial_x^2 u = 0$

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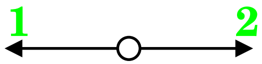
- Velocity space: 

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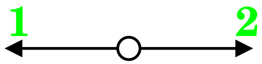
- Velocity space: 

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- Boundary conditions: $\mathsf{f}_2(t,0) = -\mathsf{f}_1(t,0) \quad \Leftrightarrow \quad u(t,0) = 0$
- Initial conditions: $\mathsf{f}_1(0,\cdot),\ \mathsf{f}_2(0,\cdot) \quad \Leftrightarrow \quad u(0,\cdot),\ \partial_t u(0,\cdot) = -\partial_x\phi(0,\cdot)$

A Convergence Result

- Reformulated LB equation $\overset{\epsilon\downarrow 0}{\rightsquigarrow}$ Target equation

$$\left.\begin{array}{ll} \text{EQ:} & \epsilon^2\tau\partial_t^2 u + \partial_t u - \tau\partial_x^2 u = 0 \\ \text{BC:} & u(\cdot,0) = 0 \quad \wedge \quad u(\cdot,L) = 0 \\ \text{IC:} & u(0,\cdot) = g \quad \wedge \quad \partial_t u(0,\cdot) = h \end{array}\right\} \quad \left\{\begin{array}{ll} \text{EQ:} & \partial_t v - \tau\partial_x^2 v = 0 \\ \text{BC:} & v(\cdot,0) = 0 \ \wedge \ v(\cdot,L) = 0 \\ \text{IC:} & v(0,\cdot) = g \end{array}\right.$$

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- **Theorem:**

$$u \xrightarrow{\epsilon\to 0} v \qquad \text{in} \quad \mathcal{C}_b\big([0,\infty),\mathcal{L}^2(0,L)\big)$$

$$\partial_t u \xrightarrow{\epsilon\to 0} \partial_t v \qquad \text{in} \quad \mathcal{C}_b\big([\theta,\infty),\mathcal{L}^2(0,L)\big) \quad \text{with} \quad \theta > 0$$

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- Compatible initialization: $h \overset{!}{=} \partial_t v(0,\cdot) = \tau\partial_x^2 v(0,\cdot) = \tau\partial_x^2 g \qquad \Rightarrow \theta = 0$

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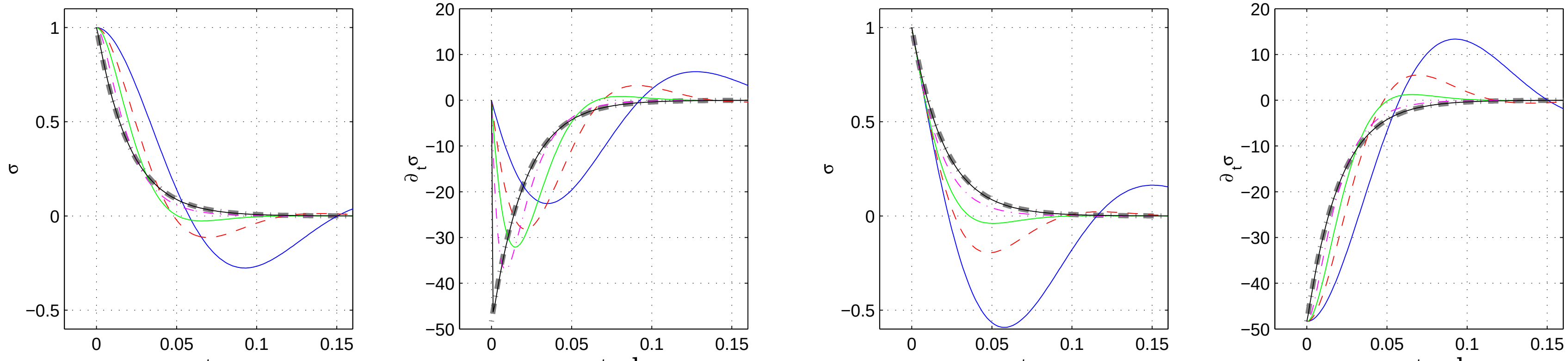
- Fourier ansatz: $u(t,x) = \sum_n \sigma_{\epsilon,n}(t) s_n(x)$, $\qquad v(t,x) = \sum_n \sigma_n(t) s_n(x)$

The Fourier coefficient functions

- Perturbed problem $\overset{\epsilon\downarrow 0}{\rightsquigarrow}$ Limit problem

$$\left.\begin{array}{ll}\text{EQ:} & \epsilon^2\tau\ddot{\sigma}_\epsilon + \dot{\sigma}_\epsilon + \lambda\sigma_\epsilon = 0 \\ \text{IC:} & \sigma_\epsilon(0) = \alpha \ \wedge \ \dot{\sigma}_\epsilon(0) = \beta\end{array}\right\} \qquad \left\{\begin{array}{ll}\text{EQ:} & \dot{\sigma} + \lambda\sigma = 0 \\ \text{IC:} & \sigma(0) = \alpha\end{array}\right.$$

- Fourier coefficients = time evolution of single-mode solution of original PDE

- 

σ_ϵ $\quad$ initial layer ON $\quad$ $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$ $\quad$ σ_ϵ $\quad$ initial layer OFF $\quad$ $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$

- Regular expansion: $\sigma_\epsilon(t) = \varsigma^{(0)}(t) + \epsilon^2\varsigma^{(2)}(t) + \ldots$
- Two-scale expansion: $\sigma_\epsilon(t) = \sigma^{(0)}(t/\epsilon^2, t) + \epsilon^2\sigma^{(2)}(t/\epsilon^2, t) + \ldots$

Regular Expansion

$$\sigma_\epsilon(t) = \varsigma^{(0)}(t) + \epsilon^2\varsigma^{(2)}(t) + \epsilon^4\varsigma^{(4)}(t) \ldots$$

- ODEs determining the asymptotic order functions:

$$\begin{array}{llll} \epsilon^0: & \dot\varsigma^{(0)} + \lambda\varsigma^{(0)} = 0 & \varsigma^{(0)}(0) = \alpha & \dot\varsigma^{(0)}(0) = \beta \\ \epsilon^2: & \dot\varsigma^{(2)} + \lambda\varsigma^{(2)} = -\tau\ddot\varsigma^{(0)} & \varsigma^{(2)}(0) = 0 & \dot\varsigma^{(2)}(0) = 0 \\ \epsilon^4: & & \vdots & \end{array}$$

Regular Expansion

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- ill-posed IVPs $\Rightarrow$ failure of regular expansion!

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- ill-posed IVPs $\Rightarrow$ failure of regular expansion!

- Way-out: $\beta \to \beta_\epsilon = \beta^{(0)} + \epsilon^2 \beta^{(2)} + \ldots$ with:

$$\beta^{(0)} = \dot\varsigma^{(0)}(0) \quad \wedge \quad \beta^{(2)} = \dot\varsigma^{(2)}(0) \quad \wedge \quad \ldots$$

- $\varsigma^{(2k)}(0) = \alpha\delta_{0k}$ given $\Rightarrow$ $\dot\varsigma^{(0)}(0), \dot\varsigma^{(2)}(0), \ldots$ *a priori* computable:

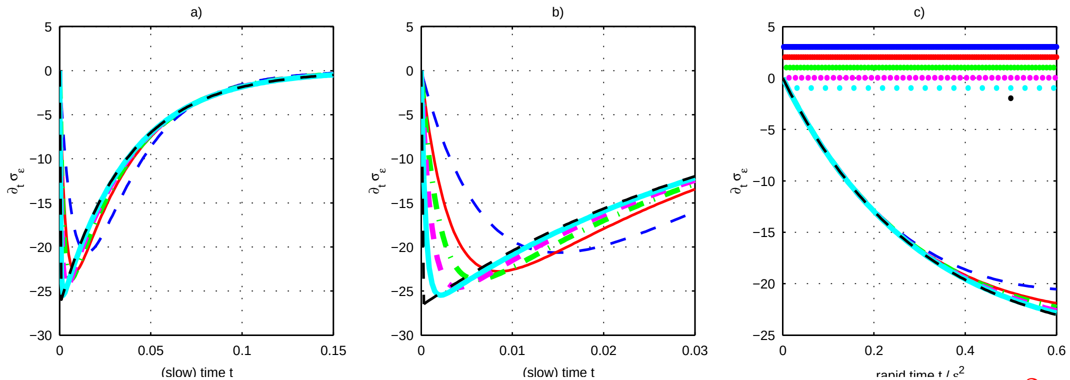
$$\dot\varsigma^{(0)}(0) = -\alpha\lambda \quad \Rightarrow \quad \ddot\varsigma^{(0)}(0) = \alpha\lambda^2 \quad \Rightarrow \quad \dot\varsigma^{(2)}(0) = -\tau\alpha\lambda^2 \quad \ldots$$

- What about arbitrary β ? $\to$ Two-scale expansion.

Two-Scale Expansion

$$\sigma_\epsilon(t) = \sigma^{(0)}(t/\epsilon^2, t) + \epsilon^2\sigma^{(2)}(t/\epsilon^2, t) + \epsilon^4\sigma^{(4)}(t/\epsilon^2, t) + \ldots$$

- As motivation consider $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$:



slow time t close-up fast time t/ϵ^2

Generally: $\phi^{(2k+2)} \leftarrow \dot\phi^{(2k)}(0),\ \dot\zeta^{(2k)}(0),\ \ddot\phi^{(2k)}$ $\zeta^{(2k)} \leftarrow \phi^{(2k)}(0),\ \ddot\zeta^{(2k-2)}$

Two-Scale Expansion

$$\sigma_\epsilon(t) = \sigma^{(0)}(t/\epsilon^2, t) + \epsilon^2\sigma^{(2)}(t/\epsilon^2, t) + \epsilon^4\sigma^{(4)}(t/\epsilon^2, t) + \dots$$

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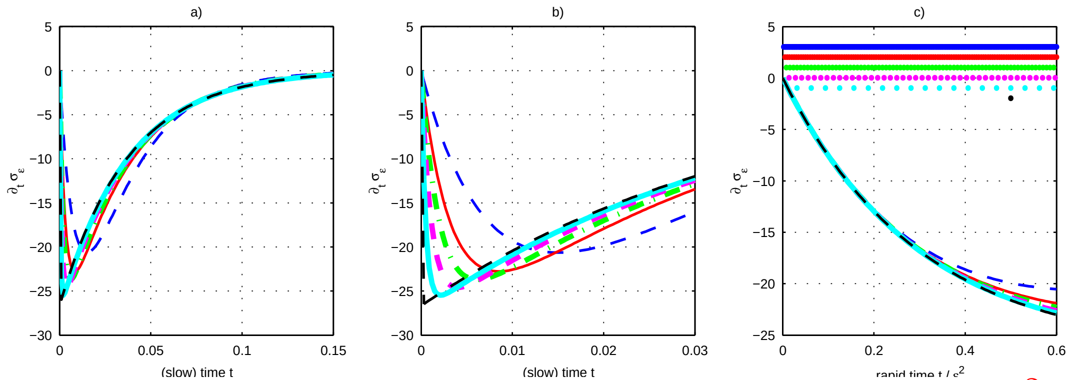

slow time t close-up fast time t/ϵ^2

- Structure of order functions: $\sigma^{(2k)}(t/\epsilon^2, t) = \underbrace{e^{-\omega t/\epsilon^2}\phi^{(2k)}(t)}_{\text{irregular}} + \underbrace{\zeta^{(2k)}(t)}_{\text{regular}}$

Two-Scale Expansion

$$\sigma_\epsilon(t) = \sigma^{(0)}(t/\epsilon^2, t) + \epsilon^2\sigma^{(2)}(t/\epsilon^2, t) + \epsilon^4\sigma^{(4)}(t/\epsilon^2, t) + \dots$$

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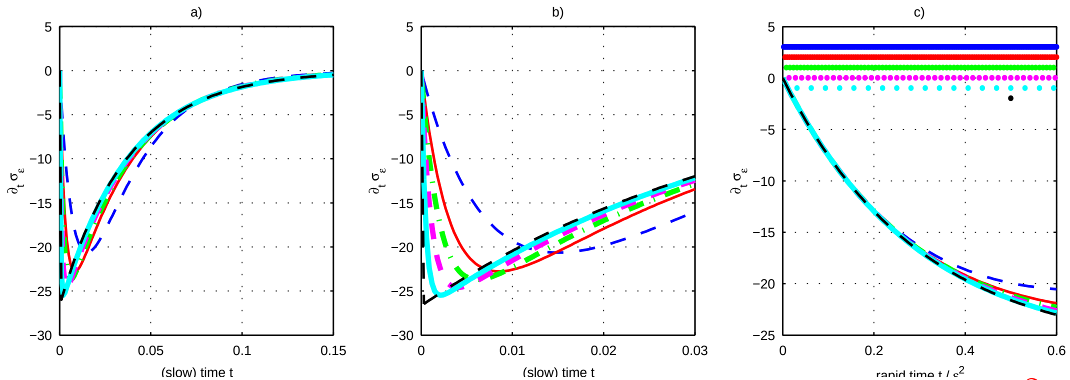


slow time t close-up fast time t/ϵ^2

- Structure of order functions: $\sigma^{(2k)}(t/\epsilon^2, t) = \underbrace{e^{-\omega t/\epsilon^2}\phi^{(2k)}(t)}_{\text{irregular}} + \underbrace{\zeta^{(2k)}(t)}_{\text{regular}}$

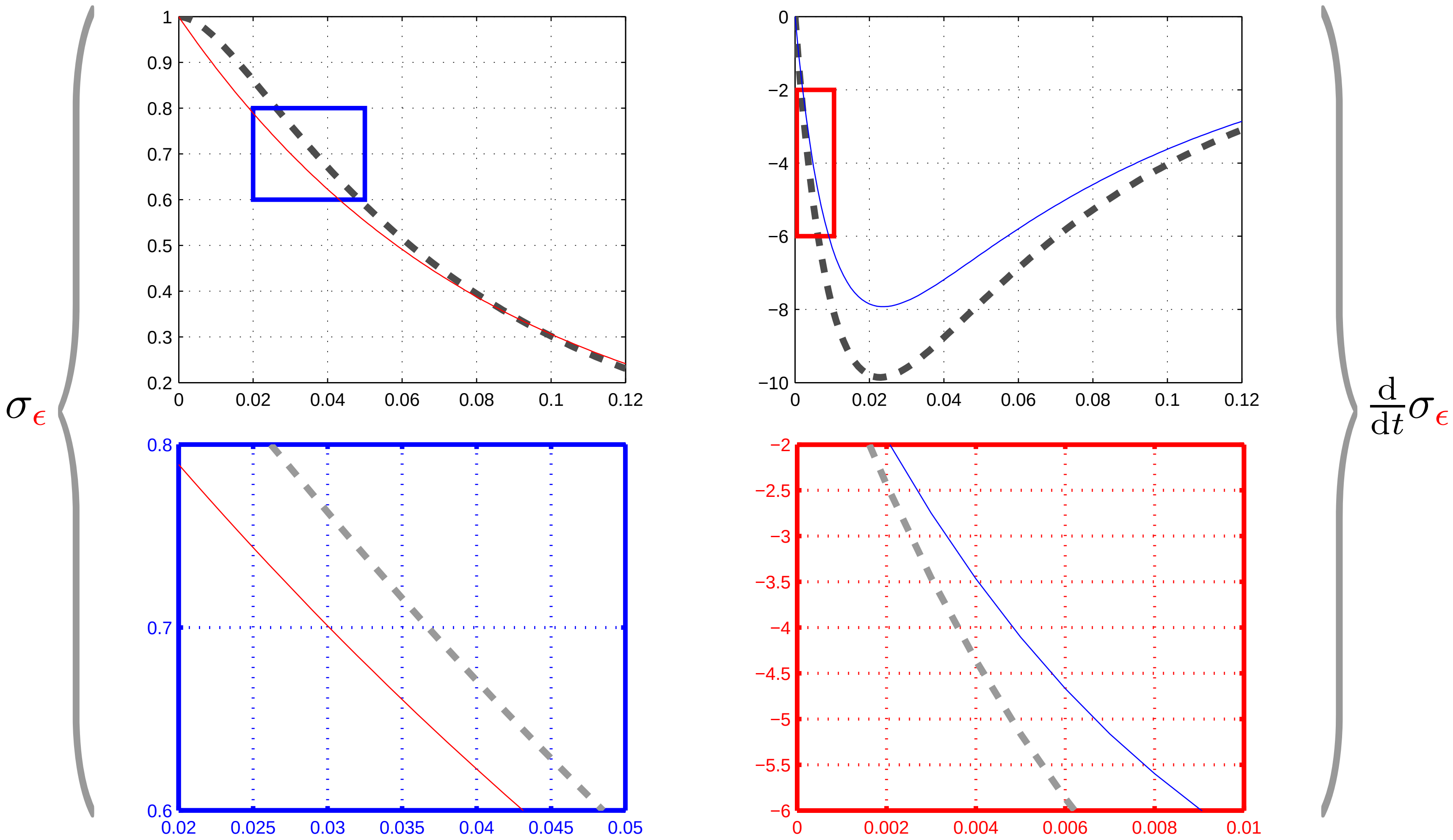
- Hierarchic ODE-system defining the asymptotic order functions:

ϵ^0:	$\phi^{(0)} \equiv 0$	$\dot\zeta^{(0)} + \lambda\zeta^{(0)} = 0$ $\zeta^{(0)}(0) = \alpha$
ϵ^2:	$\dot\phi^{(2)} - \lambda\phi^{(2)} = 0$ $\phi^{(2)}(0) = \tau\dot\zeta^{(0)}(0) - \tau\beta$	$\dot\zeta^{(2)} + \lambda\zeta^{(2)} = -\tau\ddot\zeta^{(0)}$ $\zeta^{(2)}(0) = -\phi^{(2)}(0)$

Generally: $\phi^{(2k+2)} \leftarrow \dot\phi^{(2k)}(0),\ \dot\zeta^{(2k)}(0),\ \ddot\phi^{(2k)}$ $\zeta^{(2k)} \leftarrow \phi^{(2k)}(0),\ \ddot\zeta^{(2k-2)}$

Verification of the Two-Scale Expansion

0) $\sigma_\epsilon \quad \leftrightarrow \quad \sigma^{(0)}$

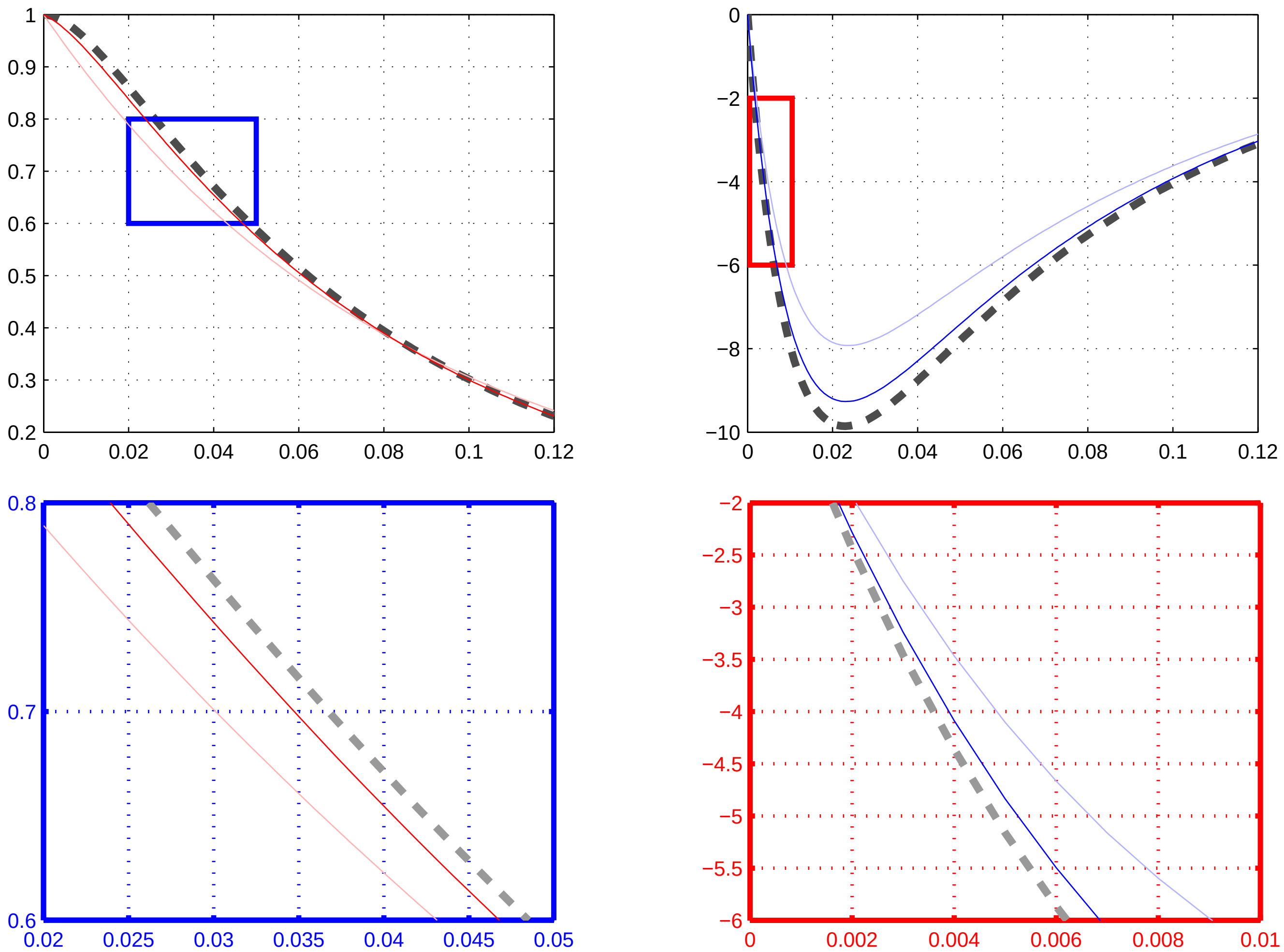


$\epsilon = 0.17, \ \tau = 0.3, \ \lambda = 4\tau\pi^2$, Caution: Differentiating the expansion for σ_ϵ does not directly yield the expansion for $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$

Verification of the Two-Scale Expansion

2) $\sigma_\epsilon \quad \leftrightarrow \quad \sigma^{(0)} + \epsilon^2 \sigma^{(2)}$

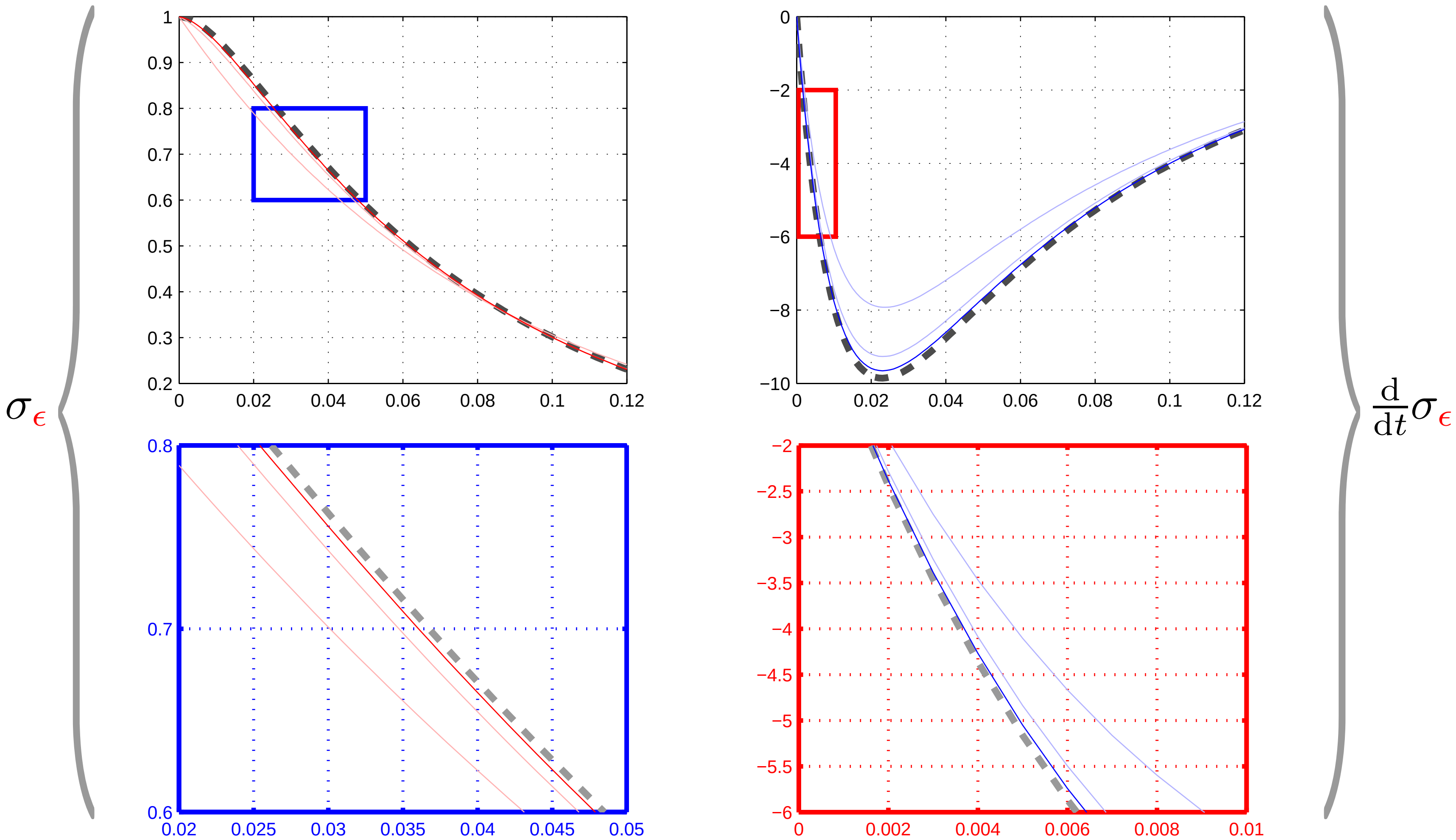
σ_ϵ



$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$

Verification of the Two-Scale Expansion

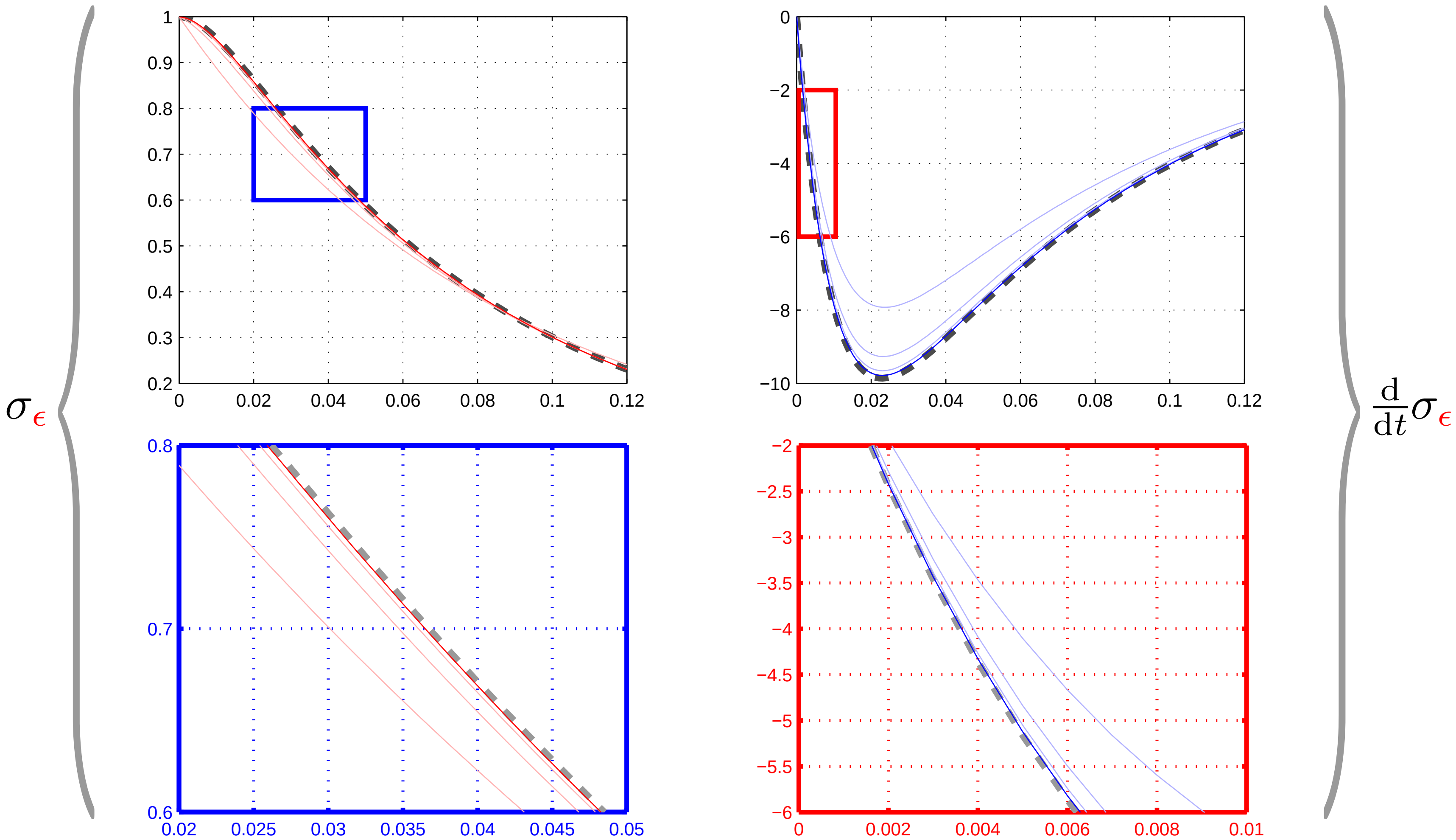
4) $\sigma_\epsilon \quad \leftrightarrow \quad \sigma^{(0)} + \epsilon^2 \sigma^{(2)} + \epsilon^4 \sigma^{(4)}$



$\epsilon = 0.17, \ \tau = 0.3, \ \lambda = 4\tau\pi^2,$ Caution: Differentiating the expansion for σ_ϵ does not directly yield the expansion for $\frac{\mathrm{d}}{\mathrm{d}t}\sigma_\epsilon$

Verification of the Two-Scale Expansion

6) $\sigma_\epsilon \quad \leftrightarrow \quad \sigma^{(0)} + \epsilon^2\sigma^{(2)} + \epsilon^4\sigma^{(4)} + \epsilon^6\sigma^{(6)}$



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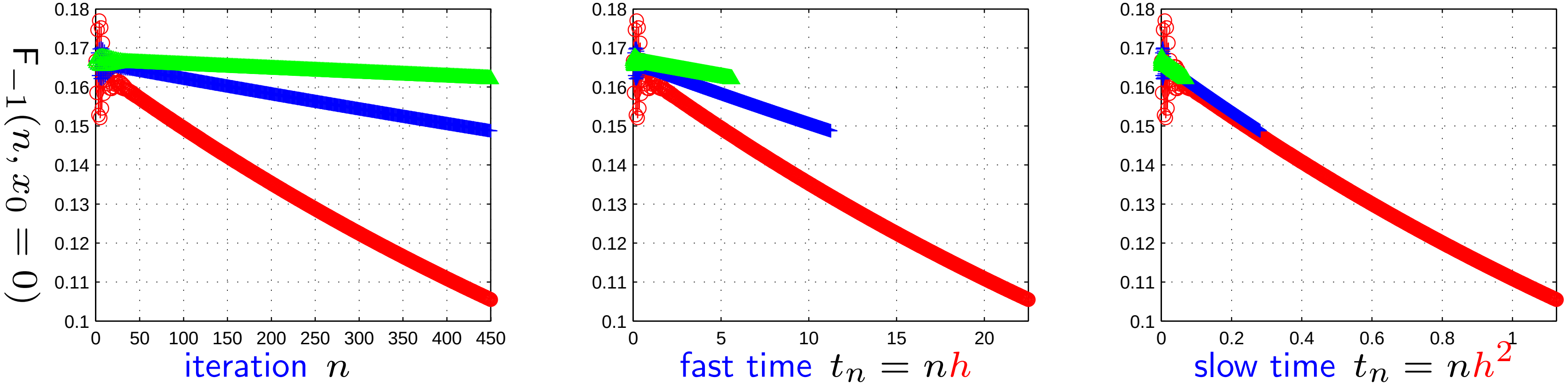
Multiple Scales of the D1P3 LB-Algorithm (ongoing research)

- D1P3 velocity space: $\mathsf{w} = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6})$ $\mathsf{s} = (-1, 0, 1)$

Initialization: $\mathsf{F}(0, x_i) = \cos(2\pi x_i / L)\mathsf{w}$, $(\nu = 0.01)$

- 

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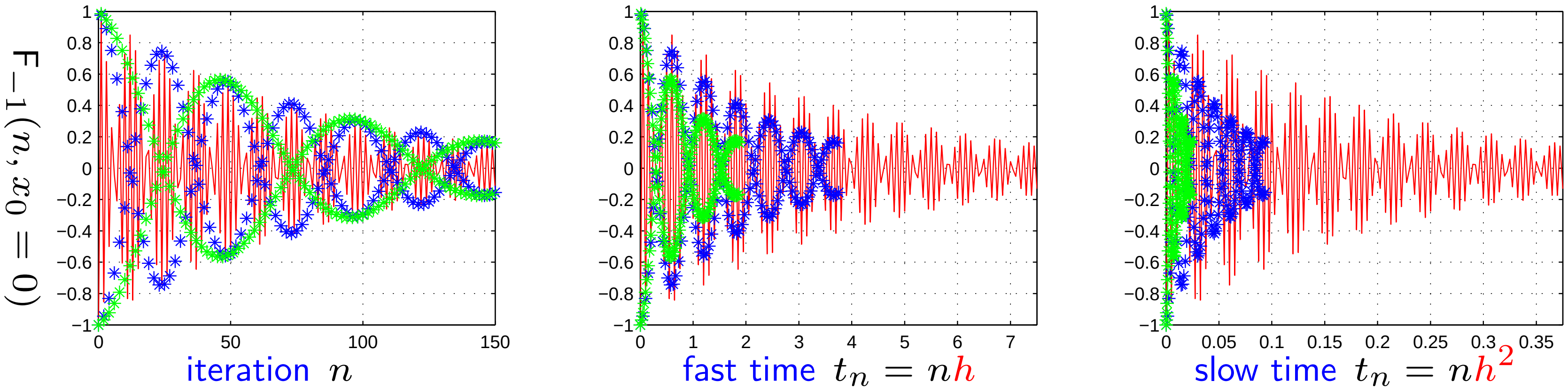
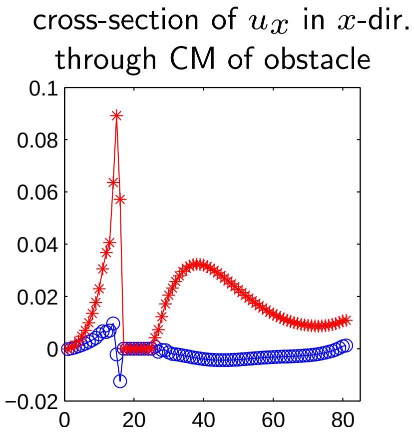
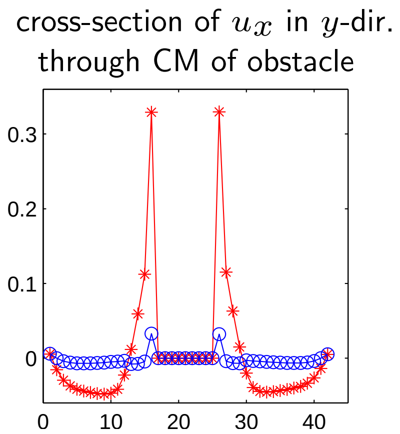
- 

periodic BC, $\theta = 3$, $\omega = (\theta\nu + 0.5)^{-1}$, red=20, blue=40, green=80 nodes

Analysis of a Boundary Layer





Channel flow around obstacle
○: Bouzidi, ∗: Bounce-back

C'est surtout grâce aux conditions de bord,
qu'un schéma numérique est précis et fort.
Finalement, elles décident de son sort.
(Si l'on va l'utiliser ou rejeter.)

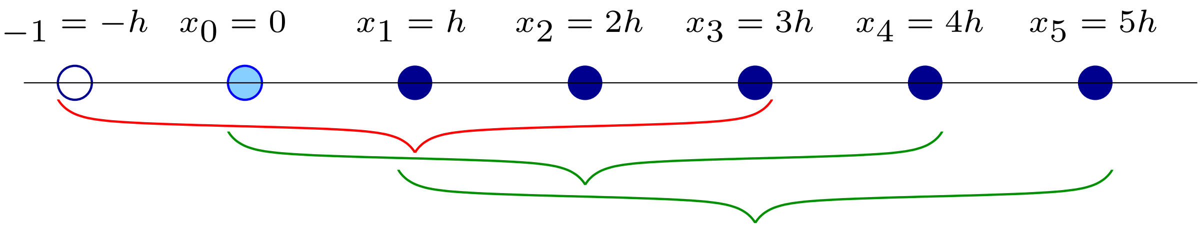
5-point stencil discretizing the 2^{nd} derivative $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$

- Poisson problem: $u(0) = u_0,\ u(L) = u_L$ and $u''(x) = f(x)$ for $x \in (0, L)$

- Discretization in the bulk:

$$\frac{1}{h^2}\underbrace{\left[a,\quad b,\quad \overset{*}{c},\quad b,\quad a\right]_h u(x)}_{au(x-2h)+bu(x-h)+cu(x)+bu(x+h)+au(x+2h)} = u''(x) + \mathrm{O}(h^4)$$

- Difficulties at the boundary:



- Modified stencil in x_1 (and anlogously in x_{N-1}) by extrapolation:

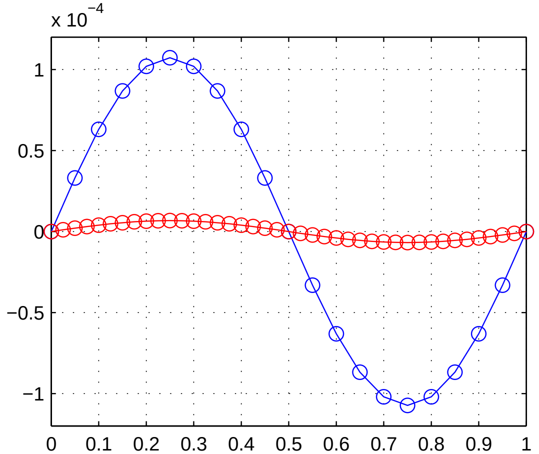
nearest-neighbor	$v_{-1} = v_0$	$\Rightarrow$	$h^{-2}[b+a,\ \overset{*}{c},\ b,\ a]v_1 = f(h)$
linear	$v_{-1} = 2v_0 - v_1$	$\Rightarrow$	$h^{-2}[b+2a,\ c \overset{*}{-} a,\ b,\ a]v_1 = f(h)$

$a = -\frac{1}{12},\ b = \frac{16}{12},\ c = -\frac{30}{12}$

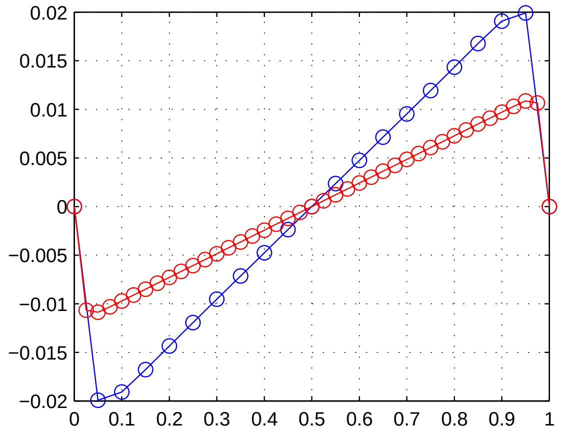
A little Experiment

	Periodic Boundary Conditions	Dirichlet Boundary Conditions
EQ:	$\partial^2 u_P = -\frac{4\pi^2}{L^2}\sin(2\pi x/L)$	$\partial^2 u_D = -\frac{4\pi^2}{L^2}\sin(2\pi x/L)$
BC:	$u_P(0) = u_P(L)$	$u_D(0) = 0 \wedge u_D(L) = 0$

- Solution: $u_P(x) = u_D(x) = u(x) = \sin(2\pi x/L)$
- Numeric solution using the 5-point stencil



error: $v_P - \hat{u}$ vs. x



error: $v_D - \hat{u}$ vs. x

- *Conclusion*: The figures suggest, that in the case of periodic boundary conditions the error can be described by smooth functions . In contrast to v_P the numeric solution of the nearest neighbor extrapolation scheme v_D reveals a strongly grid-dependent behavior (the unscaled error develops steeper and steeper jumps). Furthermore the convergence order may be significantly impaired by the boundary conditions.

Addtional condition: $\sum_i v_{P,i} = 0$ to get expected solution.

Irregular Expansion

- Ansatz: $u^{[n]} := \left(\hat{u}^{(0)} + s^{(0)}\right) + h\left(\hat{u}^{(1)} + s^{(1)}\right) + \ldots + h^n\left(\hat{u}^{(n)} + s^{(n)}\right)$

  with $\begin{cases} \text{smooth, grid-independent functions } u^{(0)}, u^{(1)}, \ldots \text{ restricted onto the grid} \\ \text{(pure) grid functions } s^{(0)}, s^{(1)}, \ldots \end{cases}$

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- Residual: $r^{[n]} := \hat{D}u^{[n]} - \varsigma = \begin{cases} u_0^{[n]} & - \quad u_0 \\ \frac{1}{h^2}[a+b,\ \overset{*}{c},\ b,\ a]u_1^{[n]} & - \quad f(h) \\ \frac{1}{h^2}[a,\ b,\ \overset{*}{c},\ b,\ a]\ u_i^{[n]} & - \quad f(ih) \qquad i \in \{2, \ldots, N-2\} \\ \frac{1}{h^2}[a,\ b,\ \overset{*}{c},\ a+b]u_{N-1}^{[n]} & - \quad f(Nh - h) \\ u_N^{[n]} & - \quad u_L \end{cases}$

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- Perform Taylor expansion of $u^{(k)}$ up to the order $n-k$.

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- BVPs demanded for $u^{(0)}, u^{(4)}$: $\begin{cases} u^{(0)}(0) = u_0 \wedge u^{(0)}(L) = u_L \wedge \partial^2 u = f \\ u^{(4)}(0) = 0 \quad \wedge u^{(4)}(L) = 0 \quad \wedge \partial^2 u = -\frac{1}{90}\partial^6 u^{(0)} \end{cases}$

Irregular Expansion (continued)

- Equations demanded for $s^{(1)}$: $$\begin{cases} h s_0^{(1)} & = & 0 \\ \frac{1}{h^2}[a+b,\ \overset{*}{c},\ b,\ a] h s_1^{(1)} & = & \frac{1}{h}\frac{1}{12}\partial u^{(0)}(0) \\ \frac{1}{h^2}[a,\ b,\ \overset{*}{c},\ b,\ a]\ h s_i^{(1)} & = & 0 \qquad i \in \{2, \ldots, N-2\} \\ \frac{1}{h^2}[a,\ b,\ \overset{*}{c},\ a+b] h s_{N-1}^{(1)} & = & -\frac{1}{h}\frac{1}{12}\partial u^{(0)}(L) \\ h s_N^{(1)} & = & 0 \end{cases}$$

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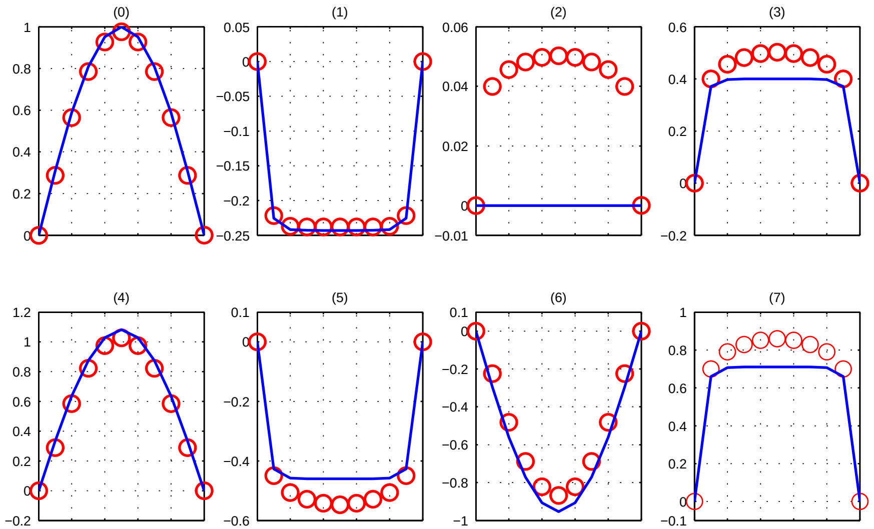
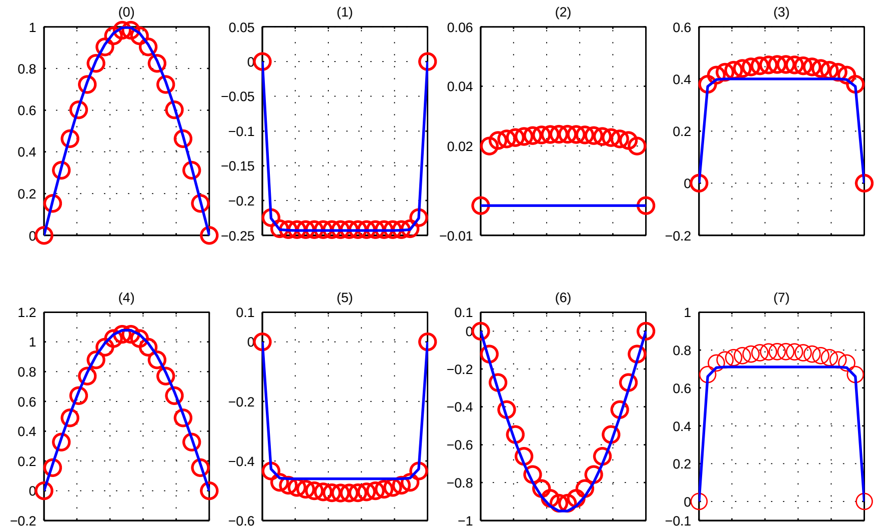
- Convergence is established in spite of negative consistency order!

Validation I

Legend:
- — n^{th} asymptotic order function
- ○ $\underbrace{\text{[numeric solution – expansion up to previous order]}}_{\text{tail of expansion}}/h^n$



11 nodes



20 nodes

$$u(x) = \sin(\pi x)$$

Validation II

Legend:
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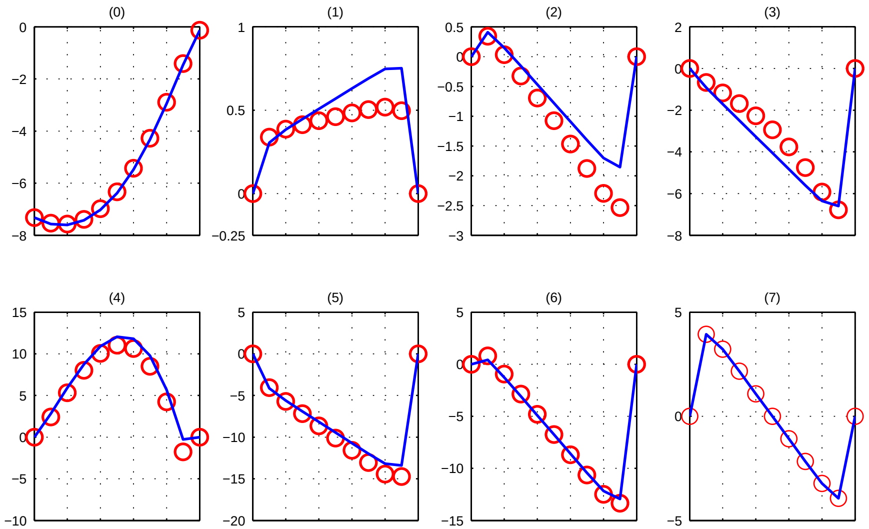

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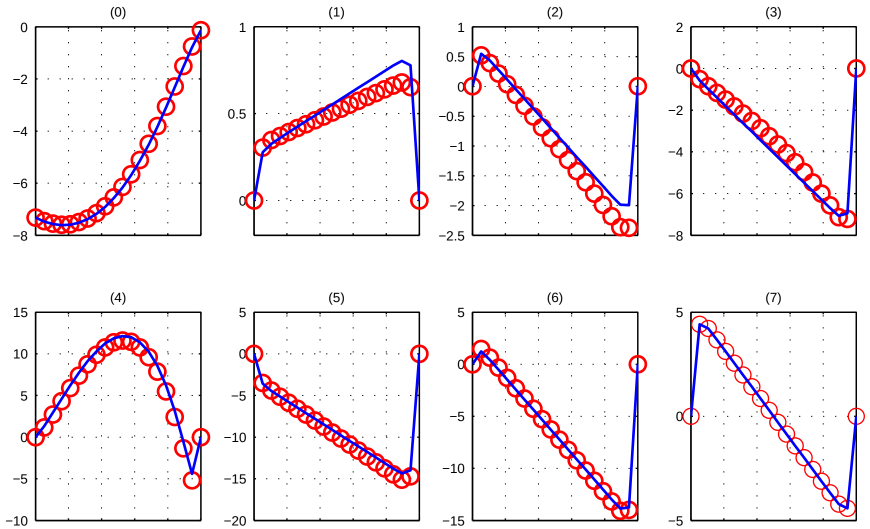
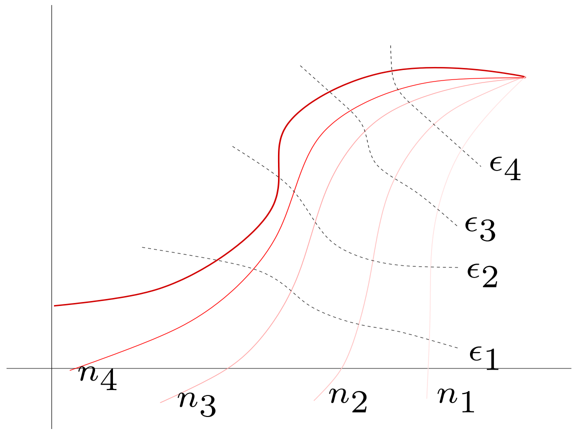


20 nodes

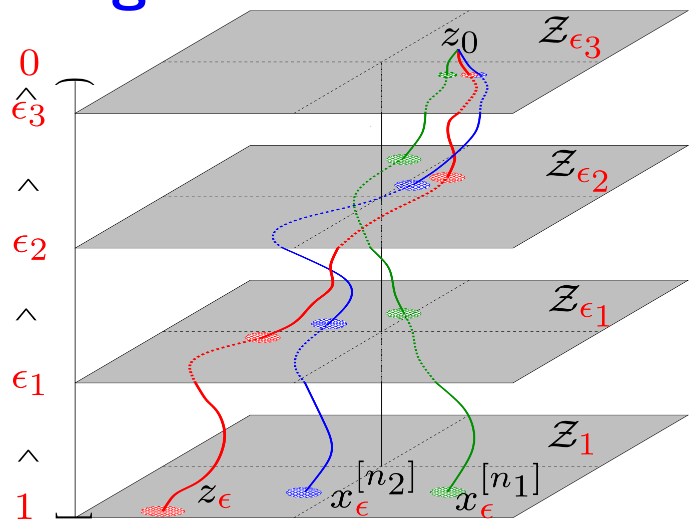
random polynomial: $u(x) = -6.667x^7 + 5.89x^6 + 2.48x^5 - 3.46x^4 + 2.69x^3 + 9.71x^2 - 3.45x - 7.32$

Approach to Convergence by Asymptotics

–General Concept–



Asymptotic Expansions and Convergence



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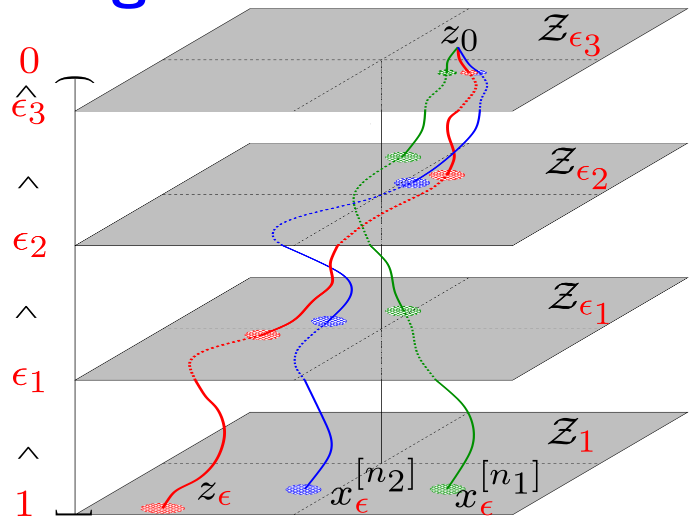

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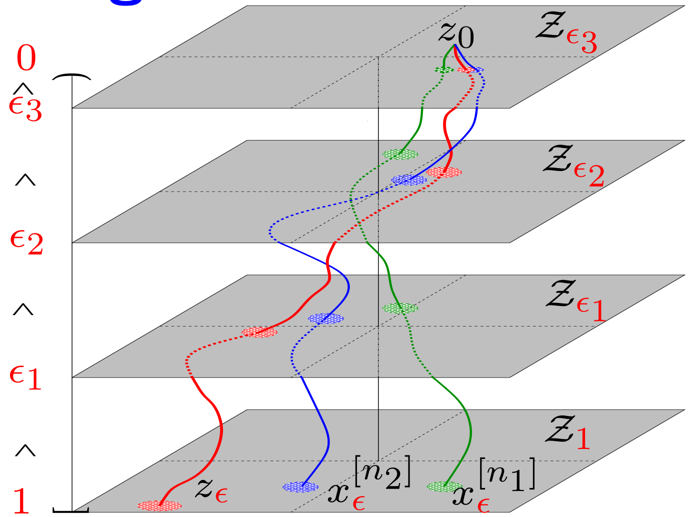

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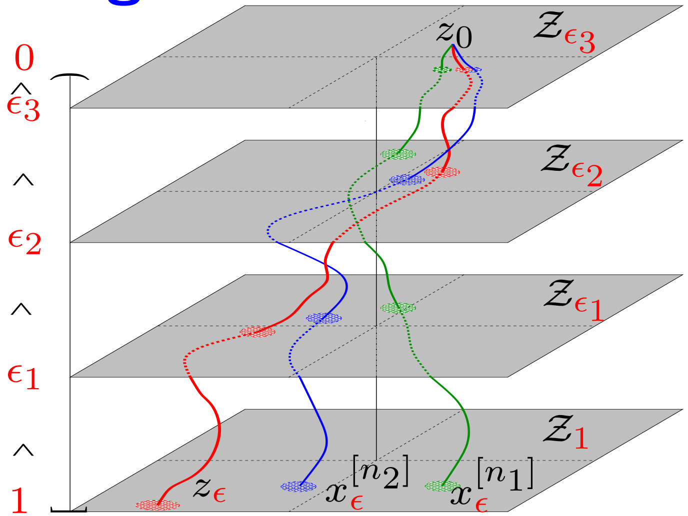

- Approximate z_ϵ in $\mathcal{X}_\epsilon = \{$special, 'simply structured' elements$\} \subset \mathcal{Z}_\epsilon$
  e.g. $\mathcal{Z}_\epsilon = \{$grid functions$\}$, $\mathcal{X}_\epsilon = \{$restrictions of smooth functions onto grid$\}$

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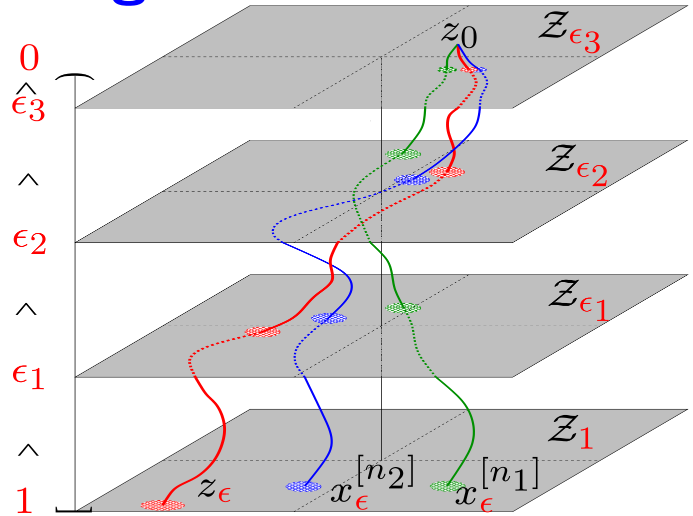

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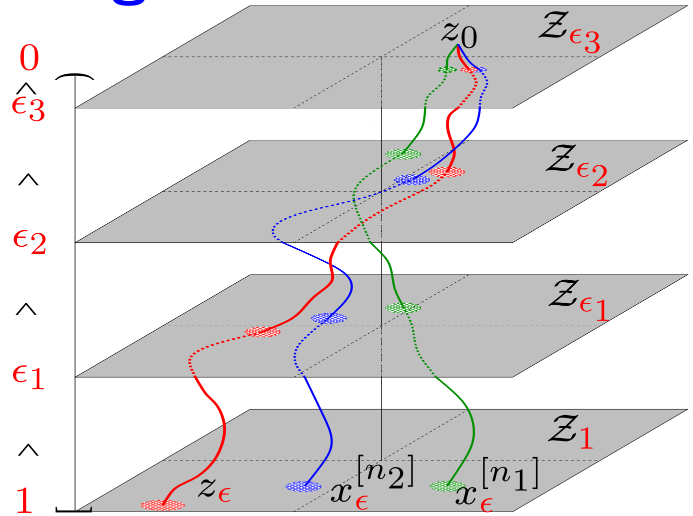

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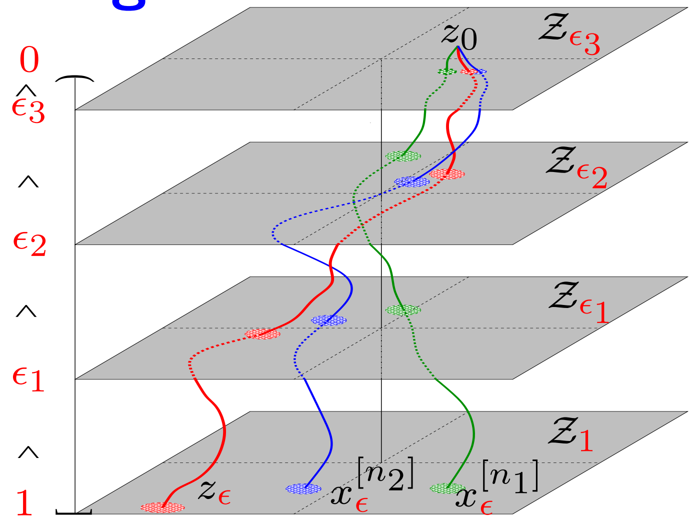

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- $$\begin{aligned} \|x_\epsilon - z_\epsilon\| &= \left\|\left(A_\epsilon^{-1} \circ A_\epsilon\right) x_\epsilon - \left(A_\epsilon^{-1} \circ A_\epsilon\right) z_\epsilon\right\| \\ &= \mathrm{Lip}_{A_\epsilon^{-1}} \big\|A_\epsilon x_\epsilon - \overbrace{A_\epsilon z_\epsilon}^{=0}\big\| = \mathrm{Lip}_{A_\epsilon^{-1}} \|r_\epsilon\| \underset{!}{\xrightarrow{\epsilon \to 0}} 0 \end{aligned}$$

Asymptotic Expansions and Convergence (continued)

- Constructing approximate solution x_{ϵ} & computing residual r_{ϵ} $\longrightarrow$ consistency analysis

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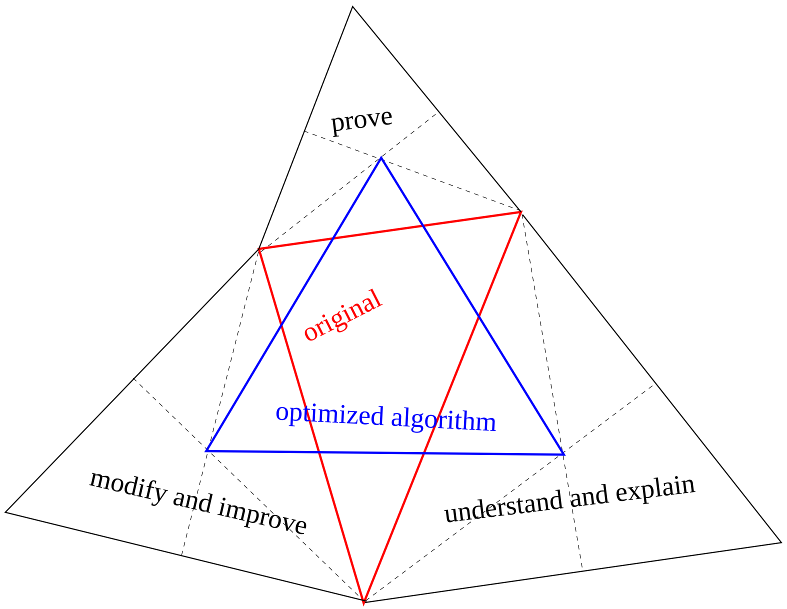
- Examples of application: convergence proofs for the D1P2- and D1P3-model with different linear equilibria and periodic boundary conditions:
  - LB-equation: $\mathcal{L}^2$-stability based on an energy estimate
  - LB-algorithm: ℓ^∞-stability using structure & positivity of evolution operator

Brief Retrospect

- Relevance of irregular expansions for
  - "continuous" problems,
  - discrete problems.
- Enables us to obtain complete understanding.

Convergence & Σummary



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- boundary layers reduce order of expansion ($\frac{1}{2}$ links help a little)

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- advantage of MRT: indication to set unused λ's to 1

Convergence

- periodic domain, smooth start or compatible initialization

$$\left\| \boldsymbol{u} - \frac{1}{h} \sum_i f_i \boldsymbol{c}_i \right\| = \mathcal{O}(h^2), \quad \left\| p - \frac{1}{h^2} \left(\sum_i f_i - 1 \right) \right\| = \mathcal{O}(h^2)$$

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- bounded domain ($\frac{1}{2}$ link), pressure and stress initialization

$$\left\| \boldsymbol{u} - \frac{1}{h}\sum_i f_i \boldsymbol{c}_i \right\| = \mathcal{O}(h^{\frac{1}{2}}), \quad \left\| p - \frac{1}{h^2}\left(\sum_i f_i - 1\right) \right\| = ?$$

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