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Numerical Methods for Partial Differential Equations
Christmas Tutorial

Exercise 1:

Consider the initial boundary value problem for the linear advection equation with piecewise constant, discontinuous initial data and periodic boundary conditions, as treated in Exercise 8 (Tutorial 4). Explain the appearance and the propagation of the pair–forming behavior of the numerical solution when using the Lax–Friedrichs scheme $v_i^0 := u_0(x_i),\ i \in I \subset \mathbb{Z};\ v_i^{n+1} = \frac{1}{2}(v_{i-1}^n + v_{i+1}^n) - \frac{a\Delta t}{2\Delta x}(v_{i+1}^n - v_{i-1}^n),\ i \in I,\ n \in \mathbb{N}$.

Prove that the exact solution of the linear advection equation satisfies the difference equation defining the Lax–Friedrichs scheme up to terms of the first order. Derive a partial differential equation (of second order) whose solution satisfies the difference equation up to terms of the second order in both variables. This equation is the so–called "modified equation" of the Lax–Friedrichs scheme. Show that any solution of the modified equation is also a solution of the advection–diffusion equation

$$\frac{\partial u}{\partial t} + a\,\frac{\partial u}{\partial x} = \frac{(\Delta x)^2}{2\Delta t}\left[1 - a^2\,\frac{(\Delta t)^2}{(\Delta x)^2}\right]\frac{\partial^2 u}{\partial x^2}.$$

Solve this equation analytically for given initial data u_0 of your choice and using periodic boundary conditions. Visualize the exact solutions of the advection equation and of the advection–diffusion equation, and compare them with the approximate solution obtained by using the Lax–Friedrichs scheme for the linear advection problem.

Exercise 2:

Explicit linear finite difference scheme applied to linear problems can be represented in the matrix form $\mathbf{v}^{n+1} = \mathbf{T}\mathbf{v}^n,\ n = 0, 1, \ldots$, with the vectors $\mathbf{v}^n$ collecting the discrete solution at time $t^n,\ n \in \mathbb{N}$, and with the specific iteration matrix $\mathbf{T}$ defined by the method and the boundary conditions. The convergence properties of the scheme are connected to the spectrum of the associated matrix.

Calculate the iteration $\mathbf{T}_{UP}, \mathbf{T}_{LF}, \mathbf{T}_{LW}$ associated to the upwind, Lax–Friedrichs and Lax–Wendroff schemes, respectively, when applied to the linear advection equation with periodic boundary conditions. Localize their spectra in the complex plane.

Write a Matlab code to calculate and visualize these spectra for different Courant numbers and different numbers of discretization points. Use the numerical results to derive ansatz formulae for the eigenvalues and calculate these analytically.

These are Bonus–exercises. Solving them can be very useful for you to understand additional aspects related to linear schemes, and can also increase your credit budget regarding the programming work. Nevertheless, these exercises should also contribute for a good holiday time.

Merry Christmas and a Happy New Year 2002!