Universität Kaiserslautern
Fachbereich Mathematik
Dr. Cristian–Aurelian Coclici

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Numerical Methods for Partial Differential Equations
Tutorial 1

Exercise 1: Consider the initial value problem for the scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(f(u)\big) &= 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

where the functions $f \in C^2(\mathbb{R})$ and $u_0 \in C^1(\mathbb{R})$ satisfy the relation

$$\exists\, I \subset \mathbb{R} \quad : \quad f''\big(u_0(x)\big)\, u_0'(x) < 0 \quad \text{for all} \quad x \in I.$$

1. Use the method of characteristics to show that the initial value problem (1) admits a classical solution $u(x,t)$ defined for $0 \le t < t_B$ with a positive time t_B;
2. Apply the theory of implicite functions to determine the "breaking time"
$$t_B = \inf_{x \in I} \left[-\frac{1}{f''\big(u_0(x)\big)\, u_0'(x)} \right] > 0.$$
3. Check the relation $u(x,t) = u_0(x - tf'(u(x,t)))$ for $0 \le t < t_B$ and use it to show that the partial differential equation $(1)_1$ is pointwise satisfied in $\mathbb{R} \times (0, t_B)$.

Exercise 2: Consider the initial value problem with the *Burgers' equation*

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\Big(\frac{u^2}{2}\Big) &= 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = u_0(x) &:= \begin{cases} 1 & \text{for} \quad x \le 0, \\ 1-x & \text{for} \quad 0 < x \le 1, \\ 0 & \text{for} \quad x > 1. \end{cases} \end{aligned} \tag{2}$$

Using the method of characteristics, show that problem (2) admits a classical solution defined for $t \in [0,1)$. Show that this solution consists of a front moving to the right and steepening until it becomes a shock at $t=1$.

Exercise 3: A simple model for the description of the vehicular traffic is given by the conservation equation for the vehicles density $\varrho = \varrho(x,t) : \mathbb{R} \times (0,\infty) \to \mathbb{R}$,

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}\left\{ 11\varrho \cdot \log \frac{142}{\varrho} \right\} = 0. \tag{3}$$

Consider the initial value problem consisting of equation (3) with the initial condition

$$\varrho(x,0) = \varrho_0(x) := \begin{cases} 10 & \text{for} \quad x \le 0 \ \text{ or } \ x \ge 60, \\ x + 10 & \text{for} \quad 0 < x \le 30, \\ 70 - x & \text{for} \quad 30 < x < 60. \end{cases}$$

Calculate the breaking time t_B and use the method of characteristics to determine ϱ at times $t = 0.5;\ 1;\ 1.5.$