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Numerical Methods for Partial Differential Equations
Tutorial 2

Exercise 4: Let $u \in L^\infty(\mathbb{R}^n \times [0,\infty))$ be a weak solution of the Cauchy problem with the conservation laws

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(f_i(u(x,t))\big) &= 0, \quad (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}^n \end{aligned} \tag{1}$$

i.e. for all $\varphi \in C_0^1(\mathbb{R}^n \times [0,\infty))$:

$$\int\limits_{\mathbb{R}^n \times (0,\infty)} \left(u\frac{\partial \varphi}{\partial t} + \sum_{i=1}^{n} f_i(u)\frac{\partial \varphi}{\partial x_i} \right)(x,t)\,dx\,dt + \int\limits_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx = 0.$$

Prove that, if u is continuously differentiable, then it satisfies the problem (1) in the classical sense. Furthermore, any weak (distributional) solution u is a classical solution of (1) in any domain $\Omega \subset \mathbb{R}^n \times (0,\infty)$ where u is C^1.

Exercise 5: Consider the initial value problem

$$\begin{aligned} u_t + u^2\,u_x &= 0 \quad \text{for} \quad (x,t) \in \Omega := \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x) := \begin{cases} 0 & \text{for} \quad x \le 0, \\ 1 & \text{for} \quad x > 0. \end{cases} \end{aligned}$$

a) Determine the solution $u(x,t)$ in $\Omega_1 := \big\{(x,t) \in \Omega \,:\, x \le 0 \ \text{ or } \ x > t\big\}$. Calculate a weak solution $u(x,t)$ containing a rarefaction wave in $\Omega \setminus \Omega_1$.

   Indication: Seek the weak solution in the form $u(x,t) = v\Big(\dfrac{x}{t}\Big)$, $(x,t) \in \Omega \setminus \Omega_1$ with an appropriate function $v \in C^1(0,1]$.

b) Construct a weak solution as a shock solution in Ω.

   Indication: The shock solution has to satisfy the Rankine–Hugoniot condition.

c) Prove that the family of functions $\big\{u_\alpha\big\}_{0<\alpha<1}$, defined by

$$u_\alpha(x,t) := \begin{cases} 0 \ , & x < \frac{1}{3}\alpha^2\,t, \ t > 0, \\ \alpha \ , & \frac{1}{3}\alpha^2\,t < x \le \alpha^2\,t, \ t > 0, \\ \sqrt{\dfrac{x}{t}} \ , & \alpha^2\,t < x < t, \ t > 0, \\ 1 \ , & 0 < t \le x, \end{cases}$$

   represents an infinite set of weak solutions, each of them containing a shock and a rarefaction wave. Construct this family by using the methods presented in the lecture.