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Numerical Methods for Partial Differential Equations
Tutorial 3

Exercise 6: Let $\varepsilon > 0$. Use the method of characteristics for solving the problem

$$\begin{aligned} u_t + u\,u_x &= 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x) := \frac{1}{\varepsilon}x \quad \text{for} \quad x \in \mathbb{R}. \end{aligned}$$

Consider then the initial value problem with the Burgers equation, where

$$u_0(x) := \begin{cases} 0 & \text{for} \quad x \le -\sqrt{\varepsilon}, \\ \frac{1}{\varepsilon}x & \text{for} \quad -\sqrt{\varepsilon} < x < \sqrt{\varepsilon}, \\ 0 & \text{for} \quad x \ge \sqrt{\varepsilon}. \end{cases}$$

Calculate the solution in this case by using the Rankine–Hugoniot conditions.

Indication: The discontinuity curves in the second case are given by $\sigma_\pm(t) = \pm\sqrt{t+\varepsilon}$.

Exercise 7: Let u be a piecewise smooth, discontinuous weak solution of

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(f(u)\big) = 0 \quad \text{in} \quad \mathbb{R} \times (0,\infty), \tag{1}$$

with a strictly convex flux function $f \in C^2(\mathbb{R})$. Let $\Sigma := \big\{(\sigma(t),t)\,|\,t > 0\big\}$ be a smooth curve in $\mathbb{R} \times [0,\infty)$ across which u is discontinuous. For $(x_0,t_0) \in \Sigma$, let $u_\ell := \lim_{\varepsilon\to 0} u(x_0-\varepsilon,t_0)$, $u_r := \lim_{\varepsilon\to 0} u(x_0+\varepsilon,t_0)$ denote the one–sided limits of u on Σ. Prove that the entropy condition

$$\big[\eta(u_\ell) - \eta(u_r)\big] \cdot \frac{d\Sigma}{dt} \le \Psi(u_\ell) - \Psi(u_r) \tag{2}$$

is satisfied for some entropy pair $(\eta(u),\Psi(u))$ with strictly convex entropy η (and with $\Psi' = \eta' f'$) if and only if the condition $\underline{\underline{u_\ell > u_r}}$ is fulfilled.

In addition, if the entropy condition (2) is satisfied for a given entropy pair (η,Ψ), then it is satisfied for any entropy pair $(\widetilde{\eta},\widetilde{\Psi})$ with $\widetilde{\eta}'' > 0$.

Indication: For the entropy pair (η,Ψ) and for fixed u_r, consider the functions $s, E_{(\eta,\Psi)} : \mathbb{R} \to \mathbb{R}$ defined by

$$s(u) := \frac{f(u) - f(u_r)}{u - u_r}, \quad E_{(\eta,\Psi)}(u) := \big[\eta(u) - \eta(u_r)\big] \cdot s(u) - \big[\Psi(u) - \Psi(u_r)\big].$$

Check the relations:

$$s' > 0 \text{ in } \mathbb{R}, \quad E_{(\eta,\Psi)}(u_\ell) \le 0, \quad E_{(\eta,\Psi)}(u_r) = 0, \quad E'_{(\eta,\Psi)}(u_r) = 0,$$

and prove the relations

$$E'_{(\eta,\Psi)}(u) = (u-u_r)(\xi - u)s'(u)\eta''(\zeta) < 0 \quad \text{for any} \quad u \ne u_r,$$

with ξ and ζ situated between u and u_r. The equivalence to be shown follows then immediately.