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Numerical Methods for Partial Differential Equations
Tutorial 4

Exercise 8 (Programming exercise):

Consider the following initial value problem for the linear advection equation

$$\begin{aligned} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} &= 0 \quad \text{for} \quad (x,t) \in \mathbb{R} \times (0,\infty), \qquad a = \text{const.} \in \mathbb{R}, \\ u(x,0) &= u_0(x) := \begin{cases} 1 & \text{for} \quad x \le 0, \\ 0 & \text{for} \quad x > 0. \end{cases} \end{aligned}$$

Establish the exact solution. If we restrict the problem to $x \in [\alpha, \beta] \subset \mathbb{R}$, then boundary conditions have to be required at $x = \alpha$ or $x = \beta$. Complete the initial value problem to an initial *boundary* value problem by prescribing boundary conditions in dependence on the sign of a. Note that the boundary and initial conditions should be compatible at $(\alpha, 0)$ and $(\beta, 0)$.

For the numerical treatment of the IBVP on $\overline{\Omega} := [\alpha, \beta] \times [0, T]$ $(T > 0)$, consider an uniform mesh of the form $\{(x_i, t^n) \mid x_i := i\Delta x,\ t^n := n\Delta t\}$, where $\Delta x, \Delta t > 0$, $i \in I := \{i_\alpha, \dots i_\beta\}$ and $n \in \{0, \dots N\}$ with integer i_α and i_β and with $N \in \mathbb{N}$. Let $\lambda := \Delta t / \Delta x > 0$.

Develop numerical codes for the approximate solution of the problem by using the following methods:

a) The *one–sided upwind scheme:*

$$v_i^0 := u_0(x_i),\ i \in I, \quad v_i^{n+1} = v_i^n - \lambda\, a \begin{cases} v_i^n - v_{i-1}^n & \text{if } a > 0, \\ v_{i+1}^n - v_i^n & \text{if } a < 0, \end{cases} \quad i \in I,\ n \in \{1, .., N\};$$

b) The *Lax–Friedrichs method:*

$$v_i^0 := u_0(x_i),\ i \in I, \quad v_i^{n+1} = \frac{1}{2}(v_{i-1}^n + v_{i+1}^n) - \frac{\lambda\, a}{2}(v_{i+1}^n - v_{i-1}^n), \quad i \in I,\ n \in \{1, .., N\};$$

c) The *Lax–Wendroff method:*

$$v_i^0 := u_0(x_i),\ i \in I,$$

$$v_i^{n+1} = v_i^n - \frac{\lambda\, a}{2}(v_{i+1}^n - v_{i-1}^n) + \frac{\lambda^2 a^2}{2}(v_{i+1}^n - 2v_i^n + v_{i-1}^n), \quad i \in I,\ n \in \{1, .., N\}.$$