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Numerical Methods for Partial Differential Equations
Tutorial 5

Exercise 9 (Programming exercise):

Consider again the scalar linear advection equation $u_t + au_x = 0$ in $(\alpha, \beta) \times (0, T)$ equipped with the initial condition $u(\cdot, 0) = u_0$ in $[\alpha, \beta]$. Imagine this equation to model the propagation of a signal, which is emitted at the left boundary $\{\alpha\} \times [0, T]$ and travels with the constant velocity a to the right. The source is modelled by a function $\varphi : [0, T] \to \mathbb{R}$ satisfying $\varphi(0) = u_0(\alpha)$.

Implement the upwind method, the Lax–Friedrichs and Lax–Wendroff schemes to approximate the propagation of the signal, using as boundary conditions at $x = \alpha$ two different signal modes, as e.g. a continously oscillating function, and a continous/discontinuous periodic pulse function.

Compare the numerical and the exact solutions.

Treat the right boundary without using information about the exact solution.

Exercise 10 (Programming exercise):

The shallow water equations can be reduced to

$$\begin{pmatrix} h \\ v \end{pmatrix}_t + \begin{pmatrix} v & h \\ 1 & v \end{pmatrix} \begin{pmatrix} h \\ v \end{pmatrix}_x = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty),$$

where $h = h(x,t)$ is the water depth and $v = v(x,t)$ is the horizontal fluid velocity. Derive a linear system by assuming that

$$h(x,t) = h_0 + \varepsilon h_1(x,t), \quad v(x,t) = v_0 + \varepsilon v_1(x,t), \quad (x,t) \in \mathbb{R} \times [0, \infty)$$

with $\varepsilon \ll 1$ and h_0, v_0 are constant, and neglecting the terms less than $\mathcal{O}(\varepsilon)$. Show that the system is strictly hyperbolic and solve it numerically by developing codes based on the left– and right–sided upwind schemes.

Test your codes in the following setting: $\alpha = -5,\ \beta = 5$,

$$h_1(x,0) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad 1 \le |x| \le 5 \end{cases}, \quad v_1(x,0) = \begin{cases} -2e^{-\frac{1}{1-4x^2}} & \text{for} \quad |x| < \frac{1}{2}, \\ 0 & \text{for} \quad \frac{1}{2} \le |x| \le 5. \end{cases}$$

Endow the problem with periodic boundary conditions and apply appropriate upwind schemes for treating the different cases

$$v_0 < -\sqrt{h_0}, \quad -\sqrt{h_0} < v_0 < \sqrt{h_0}, \quad v_0 > \sqrt{h_0}.$$