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Numerical Methods for Partial Differential Equations
Tutorial 6

Exercise 11: Consider the initial value problem with the *Burgers equation*

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(f(u)\big) &= 0 \quad \text{in} \quad \mathbb{R}\times(0,\infty), \qquad\qquad (f(u)=u^2/2) \\ u(x,0) &= u_0(x), \quad x\in\mathbb{R}. \end{aligned}$$

Take an uniform mesh of the form $\big\{(x_i,t^n)\,|\,x_i=i\Delta x,\ t^n=n\Delta t\big\}_{i\in Z\subset\mathbb{Z},\, n\in N\subset\mathbb{N}}$ with $\Delta x>0$, $\Delta t>0$ and $\lambda:=\Delta t/\Delta x$. Let $x_{i+1/2}:=(i+1/2)\Delta x$ for $i\in Z$. Implement the following numerical schemes in conservation form (set $v_i^0:=\int_{x_{i-1/2}}^{x_{i+1/2}} u_0(x)\,dx$ for $i\in Z$; the numerical fluxes $\mathbf{F}^n_{i+1/2}=\mathbf{F}(v_i^n,v_{i+1}^n)\approx\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{i+1/2},t))\,dt$ are indicated):

1. **Naive scheme**:
$$v_i^{n+1}=v_i^n-\frac{\lambda}{2}\big(f(v_{i+1}^n)-f(v_{i-1}^n)\big),\quad \mathbf{F}^N(v_i,v_{i+1})=\frac{1}{2}\big(f(v_i)+f(v_{i+1})\big);$$

2. **Upwind schemes**:
$$v_i^{n+1}=v_i^n-\lambda\begin{cases} f(v_i^n)-f(v_{i-1}^n)\ \text{(left)}, & \mathbf{F}^{LU}(v_i,v_{i+1})=f(v_i),\\ f(v_{i+1}^n)-f(v_i^n)\ \text{(right)}, & \mathbf{F}^{RU}(v_i,v_{i+1})=f(v_{i+1});\end{cases}$$

3. **Lax–Friedrichs scheme**:
$$v_i^{n+1}=\frac{v_{i-1}^n+v_{i+1}^n}{2}-\frac{\lambda}{2}\big(f(v_{i+1}^n)-f(v_{i-1}^n)\big),\ \mathbf{F}^{LF}(v_i,v_{i+1})=\frac{1}{2}\big(f(v_i)+f(v_{i+1})\big)-\frac{v_{i+1}-v_i}{2\lambda};$$

4. **Enquist–Osher scheme**:
$$v_i^{n+1}=v_i^n-\lambda\Big[\big(f^+(v_i^n)+f^-(v_{i+1}^n)\big)-\big(f^+(v_{i-1}^n)+f^-(v_i^n)\big)\Big],\mathbf{F}^{EO}(v_i,v_{i+1})=f^+(v_i)+f^-(v_{i+1}),$$
where $f^+(u):=f(0)+\int_0^u\max\{f'(s),0\}\,ds$, $f^-(u):=\int_0^u\min\{f'(s),0\}\,ds$, $u\in\mathbb{R}$.
Check the equivalent form of the numerical flux
$$\widetilde{\mathbf{F}}^{EO}(v_i,v_{i+1})=f(v_i)+f(\min\{v_{i+1},v_*\})-f(\min\{v_i,v_*\})$$
where v_* denotes the zero of f' in $\mathbb{R}$. Implement also this form of the scheme.

5. **Godunov scheme**:
$$v_i^{n+1}=v_i^n-\lambda\Big[\mathbf{F}^G(v_i^n,v_{i+1}^n)-\mathbf{F}^G(v_{i-1}^n,v_i^n)\Big]$$
with
$$\mathbf{F}^G(v_i,v_{i+1}):=\begin{cases}\max\limits_{v_{i+1}\le s\le v_i} f(s) & \text{if}\quad v_i>v_{i+1},\\ \min\limits_{v_i\le s\le v_{i+1}} f(s) & \text{if}\quad v_i\le v_{i+1}.\end{cases}$$