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Numerical Methods for Partial Differential Equations
Tutorial 7

Exercise 12: Consider the initial value problem for the scalar conservation law

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\big(f(u)\big) &= 0, \quad (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned} \tag{1}$$

with the flux function $f \in C^2(\mathbb{R})$ and the initial data $u_0 \in C^1(\mathbb{R})$. For its numerical treatment, consider the uniform grid of parameters $(\Delta x, \Delta t)$ and a 3–point conservative finite difference scheme defined by the numerical flux

$$\mathcal{H} : \mathbb{R}^3 \to \mathbb{R}, \quad \mathcal{H}(w_{-1}, w_0, w_1) = w_0 - \frac{\Delta t}{\Delta x}\Big[F(w_0, w_1) - F(w_{-1}, w_0)\Big],$$

with F satisfying the consistency condition $F(u,u) = f(u), \quad u \in \mathbb{R}$.

Prove that, if $F \in C^3(\mathbb{R}^2)$, then for any smooth solution of the Cauchy problem (**??**), the local truncation error

$$(Lu)(x,t) := \frac{u(x, t+\Delta t) - \mathcal{H}\big(u(x-\Delta x, t), u(x,t), u(x+\Delta x, t)\big)}{\Delta t}$$

satisfies the relation

$$(Lu)(x,t) = -\Delta t \frac{\partial}{\partial x}\left[\beta\big(u,\lambda\big)(x,t)\, \frac{\partial u}{\partial x}(x,t)\right] + \mathcal{O}(\Delta t^2),$$

with

$$\beta\big(u,\lambda\big)(x,t) := \frac{1}{2}\left[\frac{1}{\lambda^2}\sum_{j=-1}^{1} j^2 \frac{\partial \mathcal{H}}{\partial w_j}(u,u,u)(x,t) - \Big(f'(u)\Big)^2\right].$$

Exercise 13 (*v. Neumann stability*):

By using the Fourier transform, explore the L^2–stability of the following linear finite difference schemes:

1) the Lax–Friedrichs scheme;
2) the Lax–Wendroff scheme;
3) the Beam–Warming scheme:

$v_i^0 \in \mathbb{R}, \ i \in \mathbb{Z};$

$$v_i^{n+1} = v_i^n - \frac{\lambda\, a}{2}(3v_i^n - 4v_{i-1}^n + v_{i-2}^n) + \frac{\lambda^2 a^2}{2}(v_i^n - 2v_{i-1}^n + v_{i-2}^n), \ i \in \mathbb{Z}, \ n = 1, 2, \ldots$$

applied to the initial value problem with the linear advection equation $u_t + au_x = 0$ in $\mathbb{R} \times (0,\infty)$.