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Numerical Methods for Partial Differential Equations
Tutorial 8

Exercise 14 (Programming exercise):

For the discretization of the linear advection equation $u_t + au_x = 0$ in $(-1,1)\times(0,T)$ $(T>0)$ equipped with the initial condition $u(\cdot,0) = u_0$ in $[-1,1]$ and with periodic boundary conditions $u(-1,t) = u(1,t)$ for $t\in[0,T]$, consider a set of uniform grids $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \Delta_3$ with

$$\Delta_k := \big\{ \big(i(\Delta x)_k, n(\Delta t)_k\big) \mid i\in I_k,\ n\in\{0,...,N_k\}\big\},\quad k=0,1,2,3,$$

where $(\Delta x)_0, (\Delta t)_0 > 0$ are fixed and $(\Delta x)_k = \frac{1}{2}(\Delta x)_{k-1}$, $(\Delta t)_k = \frac{1}{2}(\Delta t)_{k-1}$ for $k=1,2,3$.

Applying the linear methods you have implemented: Upwind, Lax–Friedrichs, Enquist–Osher, Godunov, Lax–Wendroff (evtl. Beam–Warming), calculate the approximate solutions $w_k(x,t)$ on each of the different grids, and determine the corresponding global errors $e_k(x,t)$ on $[-1,1]\times[0,T]$, $k=0,1,2,3$ for each method applied.

Verify the convergence properties of the numerical solutions by checking whether the quantities

$$\sup_{0\le t\le T} \big\|e_k(\cdot,t)\big\|_V$$

tend to zero with increasing level of refinement k. As space norm $\|\cdot\|_V$ you can take

$$\big\|e_k(\cdot,t)\big\|_{L^1} = \int_{-1}^{1} |e_k(x,t)|dx,\ \big\|e_k(\cdot,t)\big\|_{L^2} = \left[\int_{-1}^{1} |e_k(x,t)|^2 dx\right]^{\frac{1}{2}},\ \big\|e_k(\cdot,t)\big\|_{L^\infty} = \max_{x\in[-1,1]} |e_k(x,t)|.$$

Give appropriate numerical evaluations for the error expressions and implement them. Examine the experimental orders of convergence

$$\alpha := \frac{\log\big(\|w_{k_1}^{ex} - w_{k_1}\|\big) - \log\big(\|w_{k_2}^{ex} - w_{k_2}\|\big)}{\log((\Delta x)_{k_1}) - \log((\Delta x)_{k_2})}$$

for the different methods by using grids with gridsizes $(\Delta x)_{k_1}, (\Delta x)_{k_2} \in \{(\Delta x)_0, (\Delta x)_1, (\Delta x)_2, (\Delta x)_3\}$. By w_k^{ex} we denote the piecewise constant projection of the exact solution w on the grid Δ_k.

Compare the experimental convergence orders with the orders of consistency of the different linear methods considered (Lax equivalence theorem).