Universität Konstanz Sommersemester 2015
Fachbereich Mathematik und Statistik
Prof. Dr. Stefan Volkwein
Sabrina Rogg

Optimierung

http://www.math.uni-konstanz.de/numerik/personen/rogg/de/teaching/

Sheet 4

Tutorial: 9th June

Exercise 11
Consider the domain

$$\Omega = \prod_{i=1}^{n} [a_i, b_i] = \{x \in \mathbb{R}^n \mid \forall i = 1, ..., n: \ a_i \leq x_i \leq b_i\,, \ a_i, b_i \in \mathbb{R}, \, a_i < b_i\}.$$

Let $f : \Omega \to \mathbb{R}$ and $f \in \mathcal{C}^0(\bar{\Omega}) \cap \mathcal{C}^1(\Omega^\circ)$, ∇f continuously expandable on $\bar{\Omega}$. Further, let $x^* \in \Omega$ be a local minimizer of f, i.e.

$$\exists\, \epsilon > 0 : \ \forall x \in B_\epsilon(x^*) \cap \Omega : \ f(x^*) \leq f(x).$$

Show that the following modified first order condition holds:

$$\forall x \in \Omega : \langle \nabla f(x^*), x - x^* \rangle \geq 0.$$

Any x^* that fulfills this condition is called *stationary point* of f.

Exercise 12
For $f : \mathbb{R}^n \to \mathbb{R}$ let L be the Lipschitz constant of the gradient ∇f. The *canonical projection* of $x \in \mathbb{R}^n$ on the closed set $\Omega = \prod_{i=1}^{n} [a_i, b_i]$ is given by $P : \mathbb{R}^n \to \Omega$,

$$\big(P(x)\big)_i := \begin{cases} a_i & \text{if } x_i \leq a_i \\ x_i & \text{if } x_i \in (a_i, b_i) \\ b_i & \text{if } x_i \geq b_i \end{cases} .$$

Further we define

$$x(\lambda) := P(x - \lambda \nabla f(x)).$$

Prove that the following modified Armijo condition holds for all $\lambda \in \left(0, \dfrac{2(1-\alpha)}{L}\right]$:

$$f(x(\lambda)) - f(x) \ \leq \ -\frac{\alpha}{\lambda} \|x - x(\lambda)\|^2.$$

Hints: The following ansatz with the fundamental theorem of calculus may be helpful:

$$f(x(\lambda)) - f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\Big(x - t\big(x - x(\lambda)\big)\Big)\mathrm{d}t.$$

You can use the following formula without proof:

$$\forall x, y \in \Omega : \ \big(y - x(\lambda)\big)^\top \big(x(\lambda) - x + \lambda \nabla f(x)\big) \ \geq \ 0.$$

Exercise 13

Let $f : \mathbb{R}^n \to \mathbb{R}$ convex and differentiable and $C \subseteq \mathbb{R}^n$ a closed and non-empty convex set. Show in the following order that:

1. $x^* \in C$ minimizes f over $C \quad \Leftrightarrow \quad \langle \nabla f(x^*), c - x^* \rangle \geq 0 \ \forall c \in C.$

   Hint: It might be helpful to use the inequality $f(b) \geq f(a) + \nabla f(a)^\top (b - a)$, $a, b \in C$, known from the lecture.

2. Let $x \in \mathbb{R}^n$ arbitrary and $c^* = P(x)$. Then, c^* is the solution to

   $$\min_{c \in C} f(c) = \min_{c \in C} \frac{1}{2}\|c - x\|^2.$$

   Prove the inequality

   $$\langle c - P(x), P(x) - x \rangle \geq 0 \quad \forall c \in C.$$

3. $x^* \in C$ minimizes f over $C \quad \Leftrightarrow \quad x^* = P(x^* - \gamma \nabla f(x^*))$ for all $\gamma \geq 0$

   Hint: It might be helpful to use (without proof) that $\langle x - P(x - \gamma \nabla f(x)), P(x - \gamma \nabla f(x)) - x + \gamma \nabla f(x) \rangle \geq 0$.