Tutorial CFD
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Tutorial 8

To be discussed on: Monday, 07.07.
Download: www.mathematik.uni-kl.de/~rutka/CFD_tutorial.html

Colourful Fluid Dynamics: Fluctuation Splitting in 2D

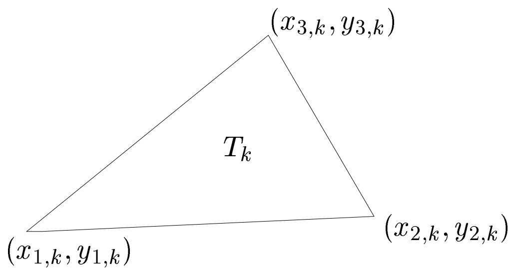
Consider a 2D linear advection equation

$$\partial_t u + a_1 \partial_x u + a_2 \partial_y u = 0$$

with a constant advection speed $\underline{a} = (a_1, a_2)$ and some initial data $u(x, y, 0) = u^0(x, y)$.

Numerical Method

Suppose domain is divided by some triangulation. Consider one triangle T_k. The function values at the vertices (1), (2) and (3) are $u_{1,k}$, $u_{2,k}$ and $u_{3,k}$ respectively. The inner normals at each triangle denote by $\underline{n}_{1,k}$, $\underline{n}_{2,k}$ and $\underline{n}_{3,k}$.



In the case if the orientation of the triangle is positive, $\underline{n}_{1,k} = (y_{2,k} - y_{3,k}, x_{3,k} - x_{2,k})$, $\underline{n}_{2,k} = (y_{3,k} - y_{1,k}, x_{1,k} - x_{3,k})$ and $\underline{n}_{3,k} = (y_{1,k} - y_{2,k}, x_{2,k} - x_{1,k})$. If the orientation of the triangle is negative, the sign of normals must be changed. (For more details see Tutorial 7.)

The update of the function u is given by

$$u_{i,k}^{n+1} = u_{i,k}^{n} + \frac{\tau}{s_{i,k}} \alpha_i \Phi + \textit{terms from other triangles} \quad , \quad i \in \{1, 2, 3\} \ ,$$

where Φ: the fluctuation at the triangle T_k. For the linear advection it is

$$\Phi = -\sum_{i=1}^{3} u_{i,k} \kappa_i \quad , \quad \kappa_i = \frac{1}{2} \underline{a} \cdot \underline{n}_{i,k} \, .$$

Coefficients α are given by (low diffusion scheme):

$$\alpha_i = \frac{\max(0, \kappa_i)}{\sum_{m=1}^{3} \max(0, \kappa_m)}$$

τ is the time step and $s_{i,k}$ is the area of the median cell, usually taken as 1/3(area of all triangles containing the vertex i).

Algorithm (Improvements welcome!)

PRE-PROCESSING

```
Load the triangulation data
     tri:  matrix corresponding to the *.tri format
     nod:  matrix corresponding to the *.nod format
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n_{nod}: number of nodes, n_{tri}: number of triangles

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Assign the initial condition

Compute area of the median cells
     Loop over all triangles:  compute area of each triangle

     Loop over all nodes.  For each node

          find in which triangles it is a vertice
          compute the area of median cell
```
S: vector with entries S_i: the area of the median cell at the vertex i

ITERATION

While $t < T_{max}$ do:

Update = zero vector of length n_{nod}

Loop over all triangles:

for ctr = 1:n_{tri}

compute $\underline{n}$, κ, α, Φ

compute the update at each node:

$$Update_{tri(ctr,1)} = Update_{tri(ctr,1)} + \frac{\tau}{S_{tri(ctr,1)}} * \alpha_1 * \Phi$$
$$Update_{tri(ctr,2)} = Update_{tri(ctr,2)} + \frac{\tau}{S_{tri(ctr,2)}} * \alpha_2 * \Phi$$
$$Update_{tri(ctr,3)} = Update_{tri(ctr,3)} + \frac{\tau}{S_{tri(ctr,3)}} * \alpha_3 * \Phi$$

end

$u = u + Update$

end

Your task

Implement the above method. Solve the linear advection equation in a square, where the triangulation is taken from the Tutorial 7 (files square.nod and square.tri). As initial data take

$$u(x, y, 0) = \begin{cases} 1 & \text{if } (x + 0.25)^2 + (y + 0.25)^2 \leq 0.05 \\ 0 & \text{else} \end{cases}$$

Choose a time step so that the method is stable.