Tutorial CFD
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Tutorial 9

To be discussed on: Monday, 14.07.
Download: www.mathematik.uni-kl.de/~rutka/CFD_tutorial.html

Finite Elements

Weak and Variational Formulations

Consider a 1D problem

$$\partial_{xx} u = f \quad \text{in} \quad \Omega = (0, 1)$$

with boundary condition $u(0) = u(1) = 0$.

Variational formulation: seek u, satisfying $u(0) = u(1) = 0$ s.t.

$$J(u) := \frac{1}{2} \int_0^1 \Big((\partial_x u)^2 - f u \Big) \, \mathrm{d}x \longrightarrow \min_u . \tag{1}$$

Weak formulation: seek u, satisfying $u(0) = u(1) = 0$ s.t. for all "sufficiently smooth" ψ we have

$$-\int_0^1 \partial_x u \, \partial_x \psi = \int_0^1 f \psi . \tag{2}$$

Remark: Under certain requirements (enought smoothness of u) all formulations can be shown to be equivalent. If u is less smooth, concretely, u and $\partial_x u$ are L^2 integrable, the last two formulations are equivalent.

Construction of Finite Element Method

First, divide the domain Ω into elements $K_1, K_2, \ldots K_N$, with nodes $x_1, x_2, \ldots x_n$. In 1D $N = n - 1$.

Discrete variational formulation: replace in (1) the unknown function u by its approximation u^h, typically written as

$$u^h = \sum_{i=1}^{n} u_i \varphi^i$$

where u_i are some coefficients and φ^i are the *basis functions*, e.g., pw-linear polynomials. If φ^i forms a *nodal basis* (i.e., $\varphi^i(x_j) = \delta_{ij}$), then the coefficients u_i are the nodal values of the function u.

Discrete weak formulation: replace in (2) the unknown function by its approximation as before, characterized by its value at n nodes. Require the relation (2) to be satisfied for n linearly independent ψ^k, $k = 1, 2, \ldots n$ from the same space as u, i.e. ψ^k can be written as

$$\psi^k = \sum_{i=1}^{n} \psi_i^k \varphi^i$$

with some coefficients ψ_i^k. Mostly $\psi^k = \varphi^k$ is used.

Your Task

Consider discretization of Ω in elements K_1, K_2, K_3, K_4 with nodes $x_1 = 0, x_2 = 0.25, x_3 = 0.5, x_4 = 0.75, x_5 = 1$. Write a finite element formulation

a) based on the variational formulation (1)

b) based on the weak formulation (2)

in the space of piecewise linear functions, with basis functions φ^k as shown in the figure below.