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Tutorial No. 3 ***Dynamical Systems: Theory and Numerics***

Return: 12am, Thursday, 29.11. metal box: Rutka/Schick (48, ground floor)

You may work in groups of up to 3 people!

Exercise 9:

Consider the 2D-system (in polar coordinates)

$$\begin{aligned} \dot{r} &= r(1-r) \\ \dot{\varphi} &= \sin^2\left(\frac{\varphi}{2}\right). \end{aligned}$$

a) Determine the critical points.

b) Compute the trajectories and draw the phase portrait (in $x-y$-space!!!).

c) Compute the ω-limitset for each trajectory.

d) Show that the point $(1,0)$ is not stable in the sense of Lyapunov.

Exercise 10:

Compute the Poincaré map for

$$\begin{aligned} \dot{x} &= \mu x + y - x\sqrt{x^2+y^2} \\ \dot{y} &= -x + \mu y - y\sqrt{x^2+y^2} \end{aligned}$$

and the transversal $L := \{(x,y) : x > 0, y = 0\}$. What is the limit cycle?

Exercise 11:

Show that for van der Pol's equation

$$\ddot{x} + x = \varepsilon(1-x^2)\dot{x}, \quad \varepsilon > 0,$$

a limit cycle exists. Hint: Poincaré-Bendixson.

Exercise 12:

Consider $\dot{x} = f(x)$, $x \in \mathbb{R}$ and a domain $D(0) \subseteq \mathbb{R}^n$ with a volume $v(0)$. The flow φ of the ODE defines

$$D(t) := \varphi(t, D(0)) = \{\varphi(t; x_0) : x_0 \in D(0)\}.$$

a) Show that for the volume $v(t)$ of $D(t)$ the following holds:

$$\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=0} = \int\limits_{D(0)} \operatorname{div} f \,\mathrm{d}x.$$

Hint: $v(t) = \int\limits_{D(0)} \det \left|\frac{\partial\varphi(t,x)}{\partial x}\right| \mathrm{d}x.$

b) *Liouville's Theorem*

Consider a time-independent Hamiltonian system

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

where $\mathcal{H}(q_1(t), p_1(t), \ldots, q_n(t), p_n(t)) = \text{const}$ is the Hamiltonian. Show that $\left.\frac{\mathrm{d}v}{\mathrm{d}t}\right|_{t=0} = 0$, i.e. Hamiltonian flows are volume preserving.