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WS 2001/02

Tutorial No. 5 ***Dynamical Systems: Theory and Numerics***

Return: 12am, Thursday, 13.12. metal box: Rutka/Schick (48, ground floor)

Exercise 17: *(Discrete Gronwall Lemma)*

Show the following: Let (p_n), (q_n) and (e_n) be positive sequences. If (e_n) satisfies the recursion

$$e_{n+1} \leq (1+q_n)e_n + p_n$$

for $n < N$, then

$$e_n \leq \left(e_0 + \sum_{j=0}^{n-1} p_j \right) \exp \left(\sum_{j=0}^{n-1} q_j \right)$$

for $n < N$. (Hint: Induction)

Exercise 18:

Consider $\dot{x} = f(t,x)$, $x(t_0) = x_0$, f L-continuous with Lipschitz-constant L. Given two numerical initial values u_0, v_0 and a grid T_h. Consider the Euler polygons u, v (according to the respective initial values) and show the estimate

$$|v_i - u_i| \leq |v_0 - u_0| e^{|t_i - t_0| L}.$$

(Hint: Use Ex. 17)

Exercise 19:

a) Show that Euler's method is consistent of order 1.

b) Determine the consistency order of the improved Euler given by

$$\psi(t,u;h) = f\left(t + \frac{h}{2}, u + \frac{h}{2} f(t,u)\right).$$

Exercise 20: *(Programming exercise)*

a) i) Solve

$$\dot{x} = t^4 x + t^4 x^4, \quad x(0) = \frac{1}{2}$$

using the Euler method and the improved Euler method on a uniform grid with various stepsizes for $t \in [0, 1]$.

ii) Plot and compare the solution with the analytic solution (Ex. 1, Tut. 1).

iii) Plot $\ln |e_h|$ versus $\ln |h|$, where h is the stepsize and e_h the error in the maximum-norm for this stepsize. What can you state about the order of convergence?

iv) Compare the computational effort (number of flops or runtime) for the two schemes to obtain the solution up to a given tolerance.

b) i) Solve the equation for the mathematical pendulum

$$\ddot{x} + \sin x = 0$$

using the Euler and the improved Euler method for $t \in [0, 40\pi]$ and a stepsize of your choice. Use the initial values $x(0) = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and $\dot{x}(0) = 0$.

ii) Plot the solutions in phase space.

iii) Monitor the Hamiltonian $\mathcal{H}$ (see Ex. 7, Tut. 2).

iv) Compute numerically the oscillation period and compare to the analytic result.