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Tutorial No. 6 ***Dynamical Systems: Theory and Numerics***
Return: 12am, Thursday, 10.01.02. Metal box: Rutka/Schick (next to 48/208)

Exercise 21:

Consider the sytem of ODEs

$$\begin{aligned}\dot{x}_1 &= \frac{\lambda_1 + \lambda_2}{2} x_1 + \frac{\lambda_1 - \lambda_2}{2} x_2 \\ \dot{x}_2 &= \frac{\lambda_1 - \lambda_2}{2} x_1 + \frac{\lambda_1 + \lambda_2}{2} x_2,\end{aligned}$$

where $\lambda_1,\ \lambda_2 < 0$ and $x_1(0) = x_{1,0},\ x_2(0) = x_{2,0}$.

a) Solve this system analytically.

b) Apply Euler's method (by hand, NOT with a computer program!!!) and compute analytically the approximation on an equidistant grid with stepsize h.

   Hint: For $\dot{x} = \lambda x \rightsquigarrow$
   $$\begin{aligned} x_1 &= x_0 + h\lambda x_0 = x_0(1 + h\lambda) \\ x_2 &= \ldots \\ x_i &= \ ? \end{aligned}$$

c) How do you have to choose the stepsize such that Euler's method converges?

d) Use $\lambda_1 = -1,\ \lambda_2 = -1000$ (this happens e.g. when modelling the kinetics of chemical reactions). Which h should one choose? How much do the solutions $x_1,\ x_2$ change on a time interval of length h?

   This phenomenon is called "stiffness" of the ODE.

Exercise 22:

The *implicit Euler method* is given by

$$\psi(x, t; h) = f(t + h, x(t + h)).$$

Then $u_{i+1} = u_i + hf(t_{i+1}, u_{i+1})$ (an implicit equation for u_{i+1}).

Reconsider part b), c) and d) of Ex. 21 for this implicit method and draw your conclusions.

Exercise 23: *(Programming exercise)*

Implement the extrapolated Euler method with adaptive order. Use a reasonable tolerance TOL and $h \sim 0.001 - 0.01$. Apply it to

a) van der Pol's equation:

$$\ddot{x} + x = \varepsilon(1 - x^2)\dot{x}$$

with $\varepsilon = 0.5,\ x(0) = 0.5,\ \dot{x}(0) = 0$ and $t_{\text{end}} = 30$.

- Plot the solution in phase-space $(x, \dot{x})$.
- Plot the iterates of the Poincaré map for the transversal $\{x = 0, y > 0\}$.

b) the Lorenz equations:

$$\begin{aligned} \dot{x}_1 &= -\sigma x_1 + \sigma x_2 \\ \dot{x}_2 &= -x_1 x_3 + r x_1 - x_2 \\ \dot{x}_3 &= x_1 x_2 - b x_3 \end{aligned}$$

with $b = \frac{8}{3},\ \sigma = 10,\ r = 28$ and the initial values $x_1(0) = -8,\ x_2(0) = 8$ and $x_3(0) = r - 1$.

- Plot the projection of the solution onto the plane $\{x_2 = 0\}$.
- Plot the projection of the solution onto the plane $\{x_3 = 0\}$.