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Tutorial No. 7 ***Dynamical Systems: Theory and Numerics***
Return: 12am, Thursday, 17.01.02. Metal box: Rutka/Schick (next to 48/208)

Exercise 24:

An ODE of the form

$$\ddot{y} + p(t)\dot{y} + q(t)y = 0$$

with $p(t),\ q(t) \in \mathcal{C}^1$ can be put in the form

$$\ddot{x} + \omega(t)x = 0$$

with $\omega(t) \in \mathcal{C}^1$, using the transformation of Liouville:

$$x(t) = y(t)e^{-\frac{1}{2}\int_{t_0}^{t} p(\tau)\mathrm{d}\tau}.$$

Does Lyapunov-stability of $(x, \dot{x}) = (0, 0)$ imply the Lyapunov-stability of $(y, \dot{y}) = (0, 0)$?

Hint: Consider the ODE $\ddot{y} - \frac{2}{t}\dot{y} + y = 0$.

Exercise 25:

Consider the two-dimensional system

$$\begin{aligned} \dot{x} &= -\frac{x}{2} - y^2 \\ \dot{y} &= -\frac{y}{4} + x^2 \end{aligned}$$

a) Check that the origin $(0, 0)$ is an equilibrium point and derive the linearization around $(0, 0)$.

b) Construct a Lyapunov function for the linear system and determine the stability of the origin.

c) Check that the Lyapunov function of 2) is an appropriate Lyapunov function for the nonlinear system.

Exercise 26: *(Symplectic Euler)*

Consider a Hamiltonian system

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

where the Hamiltonian can be written as $\mathcal{H} = T(p) + U(q)$. Then the *Symplectic Euler method* is defined by

$$\begin{aligned} q_{i+1} &= q_i + h\frac{\partial T}{\partial p}(p_i) \\ p_{i+1} &= p_i - h\frac{\partial U}{\partial q}(q_{i+1}) \end{aligned}$$

a) Determine the consistency order of the method.

b) Show that the method is invariant if p and q are interchanged and time is reversed.

c) Apply the scheme to the harmonic oscillator $\ddot{x} + x = 0$ (numerically).

Exercise 27: *(Programming exercise)*

Consider the outer solar system (see extra sheet). Use an explicit solver of your choice to compute the solution up to time $T = 200000$ (days) with step size ~ 10 (days). Monitor the Hamiltonian. All the necessary information (initial conditions, etc.) can be found on the extra sheet! Apply the symplectic Euler with fixed step size $h = 10$ days and compare.