Proper Orthogonal Decomposition for PDE Constrained Optimization

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AICES, 3 November 2011

Motivation 1: Parameter identification

- Model equations (linear, for simplicity):

$$
\begin{aligned}
-\mathrm{div}\big(c\nabla u\big) + \beta\cdot\nabla u + au &= f && \text{in } \Omega\subset\mathbb{R}^d \\
c\,\frac{\partial u}{\partial n} + qu &= g_N && \text{on } \Gamma_N\subset\Gamma=\partial\Omega \qquad (*) \\
u &= g_D && \text{on } \Gamma_D=\Gamma\setminus\Gamma_N
\end{aligned}
$$

- Problem: estimate parameters (e.g., c, β, a or q) in $(*)$ from given (perturbed) measurements u_d for the solution u on (parts of) Γ
- Mathematical formulation: (∞-dimensional) optimization problem

$$
\min \int_\Gamma \alpha\,|u-u_d|^2\,\mathrm{d}s + \kappa\,\|\mu\|^2 \quad \text{s.t.} \quad (u,\mu) \text{ solves } (*) \text{ and } \mu\in\mathcal{M}_{\mathrm{ad}}
$$

s.t. — subject to

- Numerical strategy: combine optimization methods with fast (local) rate of convergence and POD model reduction for the PDEs

Motivation 2: Optimal control of time-dependent problems

- Model problem:

$$\min \frac{1}{2}\int_\Omega |y(T)-y_T|^2\,\mathrm{d}x + \frac{\kappa}{2}\int_0^T\int_\Gamma |u|^2\,\mathrm{d}x\mathrm{d}t$$

$$\text{s.t.}\quad \begin{cases} y_t - \Delta y + f(y) & = 0 \quad \text{in } Q=(0,T)\times\Omega \\ y|_\Gamma & = u \quad \text{on } \Sigma=(0,T)\times\Gamma \\ y(0) & = y_\circ \quad \text{on } \Omega\subset\mathbb{R}^d \end{cases}$$

- Adjoint system (for gradient computation):

$$-p_t - \Delta p + f'(y)^* p = 0, \quad p|_\Gamma = 0, \quad p(T) = y_T - y(T)$$

- Optimizer: second-order methods like SQP or (semismooth) Newton
- Challenge: large-scale $\leftrightarrow$ fast/real-time optimization

Motivation 3: Closed-loop control for time-dependent PDEs

- Open-loop control:

$$\text{input } u(t) \longrightarrow \boxed{\begin{array}{l} \dot{x}(t)=f(t,x(t),u(t)) \\ x(0)=x_\circ \in \mathbb{R}^\ell \\ \text{(after spatial discretization)} \end{array}} \longrightarrow \text{output } y(t) = Cx(t) + Du(t)$$

- Closed-loop control: determine $\mathcal{F}$ with

$$u(t) = \mathcal{F}(t, y(t)) \quad \text{(feedback law)}$$

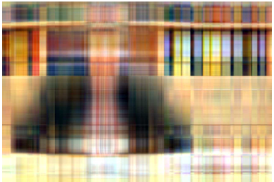
- Linear case: LQR and LQG design
- Nonlinear case: Hamilton-Jacobi-Bellman eq. (v value function)

$$v_t(t, y_\circ) + H(v_y(t, y_\circ), y_\circ) = 0 \quad \text{in } (0, T) \times \mathbb{R}^\ell$$

- Strategy: ℓ-dim. spatial approximation by, e.g., POD basis

Model reduction: POD basis containing characteristic information

0,5% der Matrixbasis −> 45% Information



Model reduction: POD basis containing characteristic information

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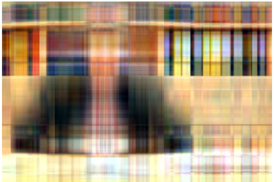

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Model reduction: POD basis containing characteristic information

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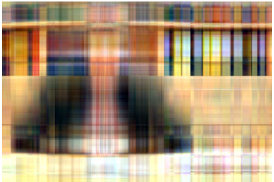

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Model reduction: POD basis containing characteristic information

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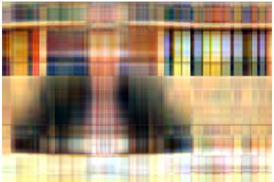

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Model reduction: POD basis containing characteristic information

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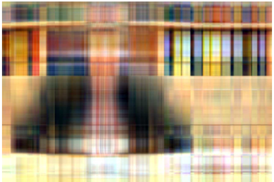

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General outline of the lecture

1.) The Proper Orthogonal Decomposition (POD) method:

- What is a POD basis?
- How can we compute the basis numerically?
- Which theory is behind?

2.) Reduced-order modelling (ROM) with POD:

- What is a POD reduced-order model?
- How can we derive a-priori error estimates?
- Can we determine an improved ROM in an adaptive way?

3.) POD suboptimal control:

- How do we apply the POD in PDE constrained optimization?
- Can we control the error?
- What can be done for feedback control?

Part 1:

The POD method

- What is a POD basis?
- How can we compute the basis numerically?
- Which theory is behind?

Outline of the first part: The POD method

- POD and singular value decomposition (SVD)
- POD method for ordinary differential equations (ODEs)
- Continuous POD method for ODEs
- POD method for partial differential equations (PDEs)
- References

POD method & SVD [Kunisch/V.'99, V.'01, V.'09]

- Given: $y_1, \ldots, y_n \in \mathbb{R}^m$; set $\mathcal{V} = \operatorname{span}\{y_1, \ldots, y_n\} \subset \mathbb{R}^m$
- Goal: Find $\ell \le \dim \mathcal{V}$ orthonormal vectors $\{\psi_i\}_{i=1}^{\ell}$ in $\mathbb{R}^m$ minimizing

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{n} \Big\| y_j - \sum_{i=1}^{\ell} (y_j^T \psi_i) \psi_i \Big\|^2$$

with the Euclidean norm $\|y\| = \sqrt{y^T y}$

- Constrained optimization:

$$\min J(\psi_1, \ldots, \psi_\ell) \quad \text{subject to} \quad \psi_i^T \psi_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Equivalent problem: Find orthonormal $\psi_1, \ldots, \psi_\ell \in \mathbb{R}^m$ maximizing

$$J^\circ(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{n} \sum_{i=1}^{\ell} \big| y_j^T \psi_i \big|^2 \quad \text{since } y_j = \sum_{i=1}^{m} (y_j^T \psi_i) \psi_i$$

Necessary optimality conditions (Part 1)

- Lagrange functional:

$$L(\psi_1,\ldots,\psi_\ell,\lambda_{11},\ldots,\lambda_{\ell\ell}) = J(\psi_1,\ldots,\psi_\ell) + \sum_{i,j=1}^{\ell} \lambda_{ij}\big(\psi_i^T\psi_j - \delta_{ij}\big)$$

  with the Kronecker symbol $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ otherwise

- Optimality conditions:

$$\begin{aligned} &\frac{\partial L}{\partial \psi_i}(\psi_1,\ldots,\psi_\ell,\lambda_{11},\ldots,\lambda_{\ell\ell}) = 0 \in \mathbb{R}^m && \text{for } i = 1,\ldots,\ell \\ &\frac{\partial L}{\partial \lambda_{ij}}(\psi_1,\ldots,\psi_\ell,\lambda_{11},\ldots,\lambda_{\ell\ell}) = 0 \in \mathbb{R} && \text{for } i,j = 1,\ldots,\ell \end{aligned}$$

Necessary optimality conditions (Part 2)

- $L(\psi_1,\ldots,\psi_\ell,\lambda_{11},\ldots,\lambda_{\ell\ell}) = J(\psi_1,\ldots,\psi_\ell) + \sum_{i,j=1}^{\ell} \lambda_{ij}\big(\psi_i^T\psi_j - \delta_{ij}\big)$
- $\dfrac{\partial L}{\partial \psi_i} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{n} y_j(y_j^T\psi_i) = \lambda_{ii}\psi_i$ and $\lambda_{ij} = 0$ for $i \neq j$
- $\dfrac{\partial L}{\partial \lambda_{ij}} = 0 \quad \Leftrightarrow \quad \psi_i^T\psi_j = \delta_{ij}$
- Setting $\lambda_i = \lambda_{ii}$ and $Y = [y_1,\ldots,y_n] \in \mathbb{R}^{m\times n}$ we have

  $$YY^T\psi_i = \lambda_i\psi_i \quad \text{for } i = 1,\ldots,\ell$$

  i.e., necessary optimality conditions are given by a symmetric $m \times m$ eigenvalue problem
- Here: necessary optimality conditions are already sufficient

Computation of the POD basis (Part 1)

- Optimality conditions: $YY^T\psi_i = \lambda_i\psi_i$ for $i = 1, \ldots, \ell$
- Solution by SVD for $Y \in \mathbb{R}^{m\times n}$: $d = \text{rank } Y$, $\sigma_1 \geq \ldots \geq \sigma_d > 0$, $U = [u_1, \ldots, u_m] \in \mathbb{R}^{m\times m}$ und $V = [v_1, \ldots, v_n] \in \mathbb{R}^{n\times n}$ orthogonal with

$$U^T YV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \Sigma \in \mathbb{R}^{m\times n}$$

  where $D = \operatorname{diag}(\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{d\times d}$. Moreover, for $1 \leq i \leq d$

$$Yv_i = \sigma_i u_i, \;\; Y^T u_i = \sigma_i v_i, \;\; YY^T u_i = \sigma_i^2 u_i, \;\; Y^T Y v_i = \sigma_i^2 v_i$$

- POD basis: $\psi_i = u_i$ and $\lambda_i = \sigma_i^2 > 0$ for $i = 1, \ldots, \ell \leq d = \dim \mathcal{V}$ with $\mathcal{V} = \text{span}\,\{y_1, \ldots, y_n\}$

Computation of the POD basis (Part 2)

- Data ensemble: $\mathcal{V} = \text{span}\,\{y_1, \ldots, y_n\} \subset \mathbb{R}^m$ and $d = \dim \mathcal{V}$
  POD basis of rank ℓ: $\psi_i = u_i$ and $\lambda_i = \sigma_i^2 > 0$ for $i = 1, \ldots, \ell \le d$
- Three choices to compute the ψ_i's
  SVD for $Y \in \mathbb{R}^{m\times n}$: $Yv_i = \sigma_i u_i$
  EVD for $YY^T \in \mathbb{R}^{m\times m}$: $YY^T u_i = \sigma_i^2 u_i$ (if $m \ll n$)
  EVD for $Y^T Y \in \mathbb{R}^{n\times n}$: $Y^T Y v_i = \sigma_i^2 v_i$ and $u_i = \frac{1}{\sigma_i} Y v_i$ (if $m \gg n$)
- Essential error formula for the POD basis of rank ℓ:

$$J(\psi_1, \ldots, \psi_\ell) = \sum_{j=1}^{n} \Big\| y_j - \sum_{i=1}^{\ell} (y_j^T \psi_i)\psi_i \Big\|^2 = \sum_{i=\ell+1}^{d} \lambda_i$$

Computation of the POD basis (Part 3)

- Essential error formula for the POD basis of rank ℓ:

$$J(\psi_1,\ldots,\psi_\ell)=\sum_{j=1}^{n}\Big\|y_j-\sum_{i=1}^{\ell}(y_j^T\psi_i)\psi_i\Big\|^2=\sum_{i=\ell+1}^{d}\lambda_i$$

- $YY^T\psi_i=\lambda_i\psi_i$, $1\le i\le\ell$, and $YY^T\psi_i=\sum_{j=1}^{n}(y_j^T\psi_i)y_j$ give

$$\lambda_i=\lambda_i\psi_i^T\psi_i=\big(YY^T\psi_i\big)^T\psi_i=\Big(\sum_{j=1}^{n}(y_j^T\psi_i)y_j\Big)^T\psi_i=\sum_{j=1}^{n}\big|y_j^T\psi_i\big|^2$$

- $y_j=\sum_{i=1}^{d}(y_j^T\psi_i)\psi_i$, $j=1,\ldots,m$, and $\psi_i^T\psi_j=\delta_{ij}$ imply

$$\sum_{j=1}^{n}\Big\|y_j-\sum_{i=1}^{\ell}(y_j^T\psi_i)\psi_i\Big\|^2=\sum_{j=1}^{n}\sum_{i=\ell+1}^{d}\big|y_j^T\psi_i\big|^2=\sum_{i=\ell+1}^{d}\lambda_i$$

Outline of the first part: The POD method

- POD and SVD
- **POD method for ODEs**
- Continuous POD method for ODEs
- POD method for PDEs
- References

POD method for ODEs

- Nonlinear dynamical system in $\mathbb{R}^m$:

$$\dot{y}(t) = f(t, y(t)) \text{ for } t \in (0, T) \quad \text{and} \quad y(0) = y_\circ$$

with given $y_0 \in \mathbb{R}^m$ and $f : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$

- Time grid: $0 \le t_1 < t_2 < \ldots t_n \le T$, $\delta t_j = t_j - t_{j-1}$ for $2 \le j \le n$
- Available or known snapshots: $y_j = y(t_j)$, $1 \le j \le n$
- Snapshot ensemble: $\mathcal{V} = \mathrm{span}\,\{y_1, \ldots, y_n\}$, $d = \dim \mathcal{V} \le n$
- POD basis of rank $\ell < d$: with weights $\alpha_j \ge 0$

$$\min \sum_{j=1}^{n} \alpha_j \Big\| y_j - \sum_{i=1}^{\ell} \big(y_j^T \psi_i\big) \psi_i \Big\|^2 \quad \text{s.t.} \quad \psi_i^T \psi_j = \delta_{ij}$$

Computation of the POD basis

- EVD for linear and symmetric $\mathcal{R}^n$ in ODE space $\mathbb{R}^m$:

$$\mathcal{R}^n u_i = \sum_{j=1}^{n} \alpha_j \, y_j \left(y_j^T u_i \right) = \sigma_i^2 u_i \qquad \left(YY^T u_i = \sigma_i^2 u_i \right)$$

and set $\lambda_i = \sigma_i^2$, $\psi_i = u_i$

- EVD for linear and symmetric $\mathcal{K}^n = \left(\left(\sqrt{\alpha_i \alpha_j} \, y_j^T y_i \right) \right)$ in $\mathbb{R}^n$:

$$\mathcal{K}^n v_i = \sigma_i^2 v_i \qquad \left(Y^T Y v_i = \sigma_i^2 v_i \right)$$

and set $\lambda_i = \sigma_i^2$, $\psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} \sqrt{\alpha_j} \left(v_i \right)_j y_j$

- Error formula for the POD basis of rank ℓ:

$$\sum_{j=1}^{n} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \left(y_j^T \psi_i \right) \psi_i \right\|^2 = \sum_{i=\ell+1}^{d} \lambda_i$$

Outline of the first part: The POD method

- POD and SVD
- POD method for ODEs
- **Continuous POD method for ODEs**
- POD method for PDEs
- References

Continuous POD method [Kunisch/V.'02, Henri'04, V.'09, Chapelle et al.'11]

- Snapshots: $y(t)$ for all $t \in [0, T]$
- Snapshot ensemble: $\mathcal{V} = \{y(t) \,|\, t \in [0, T]\}$, $d = \dim \mathcal{V} \leq \infty$
- POD basis of rank $\ell < d$:

$$\min \int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \big(y(t)^T \psi_i\big)\, \psi_i \Big\|^2 \mathrm{d}t \quad \text{s.t.} \quad \psi_i^T \psi_j = \delta_{ij}$$

- Optimality conditions: EVP for linear, symmetric, compact $\mathcal{R}$

$$\mathcal{R}\psi_i = \int_0^T \big(\psi_i^T y(t)\big)\, y(t)\,\mathrm{d}t = \lambda_i \psi_i \quad \text{for } i \in \mathbb{N}$$

- Error for the POD basis of rank ℓ:

$$\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \big(\psi_i^T y(t)\big)\, \psi_i \Big\|^2 \mathrm{d}t = \sum_{i=\ell+1}^{d} \lambda_i$$

Relationship between 'discrete' and continuous POD

- Operators $\mathcal{R}^n$ and $\mathcal{R}$:

$$\mathcal{R}^n\psi = \sum_{j=1}^{n} \alpha_j \left(\psi^T y(t_j)\right) y(t_j)$$

$$\mathcal{R}\psi = \int_0^T \left(\psi^T y(t)\right) y(t)\,\mathrm{d}t$$

- Operator convergence of $\mathcal{R}^n - \mathcal{R}$: y smooth and appropriate α_j's
- Perturbation theory [Kato'80]: $(\lambda_i^n, \psi_i^n) \stackrel{n\to\infty}{\longrightarrow} (\lambda_i, \psi_i)$ for $1 \le i \le \ell$
- Choice of the weights α_j?: ensure convergence $\mathcal{R}^n \stackrel{n\to\infty}{\longrightarrow} \mathcal{R}$

Outline of the first part: The POD method

- POD and SVD
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- Continuous POD method for ODEs
- **POD method for PDEs**
- References

POD method for PDEs

- Heat equation (for instance):

$$\begin{aligned} y_t - \Delta y &= f && \text{in } Q = (0,T) \times \Omega \\ \frac{\partial y}{\partial n} &= g && \text{on } \Sigma = (0,T) \times \Gamma \\ y(0) &= y_\circ && \text{in } \Omega \subset \mathbb{R}^l \end{aligned}$$

- Variational formulation: for $V = H^1(\Omega)$, f.a.a. $t \in [0,T]$

$$\int_\Omega y_t(t)\varphi + \nabla y(t) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(t)\varphi \,\mathrm{d}x + \int_\Gamma g(t)\varphi \,\mathrm{d}s \quad \forall \varphi \in V$$

- FE Galerkin: $y^m(t) \in V^m = \text{span}\,\{\varphi_1, \ldots, \varphi_m\}$, f.a.a. $t \in [0,T]$

$$\int_\Omega y_t^m(t)\varphi + \nabla y^m(t) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f(t)\varphi \,\mathrm{d}x + \int_\Gamma g(t)\varphi \,\mathrm{d}s \; \forall \varphi \in V^m$$

POD basis computation

- Time grid: $0 \le t_1 < t_2 < \ldots t_n \le T$, $\delta t = t_j - t_{j-1}$ for $2 \le j \le n$
- FE snapshots: $y_j = y^m(t_j) \in V^m$, $1 \le j \le n$
- Inner product: $\langle u, v\rangle = \int_\Omega uv\,\mathrm{d}x$ or $\langle u, v\rangle = \int_\Omega uv + \nabla u \cdot \nabla v\,\mathrm{d}x$
- Sizes: # FE's $\gg$ # time instances, i.e., $m \gg n$
- Computation of the correlation $\mathcal{K}^n$: $\alpha_j = \mathcal{O}(\delta t)$ $(1 < j < n)$

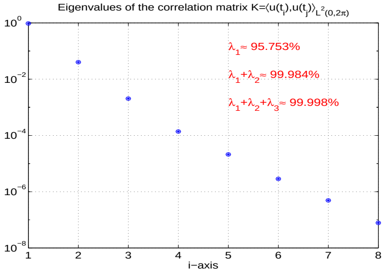
$$\sqrt{\alpha_i\alpha_j}\,\langle y_j^m, y_i^m\rangle = \sqrt{\alpha_i\alpha_j}\sum_{k,l=1}^{n} Y_{ik}Y_{jl}\langle\varphi_l, \varphi_k\rangle = \left(DY^TMYD\right)_{ij} =: \mathcal{K}_{ij}^n$$

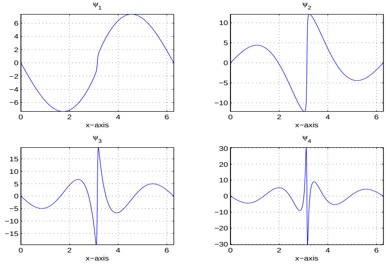
  with $M_{ij} = \langle\varphi_j, \varphi_i\rangle$ (mass or stiffness matrix) and $D = \mathrm{diag}\left(\sqrt{\alpha_i}\right)$
- POD basis: $\mathcal{K}^n v_i = \lambda_i v_i$ and $\psi_i = \frac{1}{\sqrt{\lambda_i}} YDv_i$

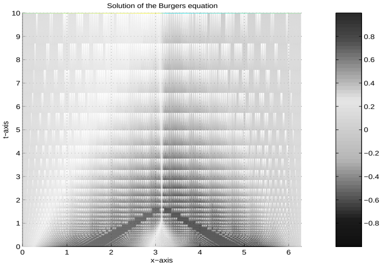
Numerical example: Burgers equation

$$
\begin{aligned}
y_t - \nu y_{xx} + y y_x &= f && \text{in } Q = (0,T) \times \Omega \\
y(\cdot, 0 = y(\cdot, 1) &= 0 && \text{on } (0,T) \\
y(0, \cdot) &= y_\circ && \text{in } \Omega = (0, 2\pi) \subset \mathbb{R}
\end{aligned}
$$

- $y_\circ(x) = \sin(x)$ and $\nu = 0.01$
- 1258 finite elements
- Time integration with Matlab's `ode15s`
- Snapshots $\mathcal{V} = \operatorname{span}\{y(t_1), \ldots, y(t_{100})\}$



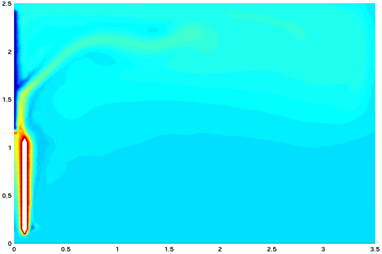


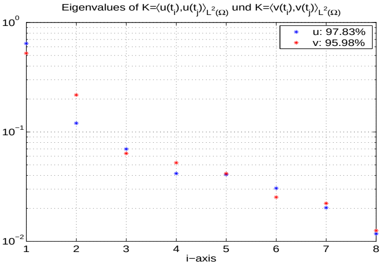


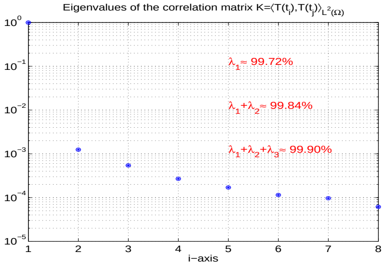
Numerical example: Energy transport (Boussinesq)

$$
\begin{aligned}
u_t + uu_x + vu_y + p_x &= \nu \Delta u && \text{in } Q\\
v_t + uv_x + vv_y + p_y &= \nu \Delta v + \beta\theta && \text{in } Q\\
u_x + v_y &= 0 && \text{in } Q\\
\theta_t + u\theta_x + v\theta_y &= \alpha \Delta\theta && \text{in } Q
\end{aligned}
$$

- $\alpha = 10^{-5}$, $\beta = 10^{-2}$, $\nu = 10^{-4}$
- 4×3512 finite elements (Femlab)
- Time integration with Matlab's ode15s
- Snapshots at $t_1, \ldots, t_{21}$ for u, v and θ







POD for λ-ω systems [Müller/V.'06]

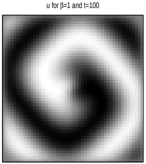
- PDEs: $s = u^2 + v^2$, $\lambda(s) = 1 - s$, $\omega(s) = -\beta s$

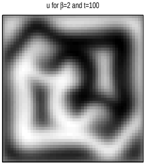
$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \lambda(s) & -\omega(s) \\ \omega(s) & \lambda(s) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \sigma \Delta u \\ \sigma \Delta v \end{pmatrix}$$

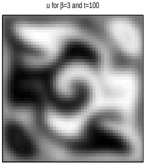
- Homogeneous boundary conditions:

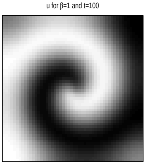
$$u = v = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0$$

- Initial conditions: $u_\circ(x_1, x_2) = x_2 - 0.5$, $v_\circ(x_1, x_2) = (x_1 - 0.5)/2$







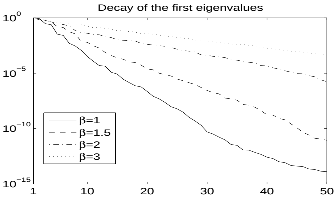


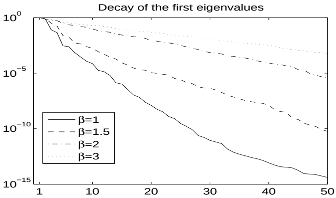
POD basis for λ-ω systems

- Offsets: $\bar{u}(x) = \frac{1}{n}\sum_{j=1}^{n} u(t_j, x)$ or $\bar{u} \equiv 0$
- Snapshots: $\hat{u}_j(x) = u(t_j, x) - \bar{u}(x)$ for $1 \leq j \leq n$
- POD eigenvalue problem: $\langle u, v\rangle = \int_\Omega uv \,\mathrm{d}x$

$$\mathcal{K}^n v_i = \lambda v_i,\ 1 \leq i \leq \ell, \quad \text{with } \mathcal{K}^n_{ij} = \sqrt{\alpha_i\alpha_j}\int_\Omega \hat{u}_j(x)\hat{u}_i(x)\,\mathrm{d}x$$

- POD basis computation: $\psi_i = \frac{1}{\sqrt{\lambda_i}}\sum_{j=1}^{n}\sqrt{\alpha_j}(v_i)_j\hat{u}_j$

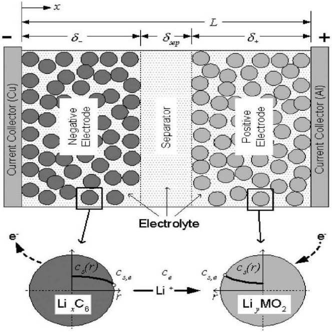




Elliptic-parabolic systems [Lass/V.'11]

- Elliptic-parabolic systems: $T = 1$, $\Omega = (a, b)$

$$
\begin{aligned}
y_t - \nabla \cdot (c_1 \nabla y) - \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q = (0, T) \times \Omega \\
-\nabla \cdot (c_2 \nabla p) - \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q \\
-\nabla \cdot (c_3 \nabla q) + \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q
\end{aligned}
$$



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-\nabla \cdot (c_3 \nabla q) + \mathcal{N}(y, p, q; \mu) &= 0 \qquad \text{in } Q
\end{aligned}
$$

- Parameter-dependent nonlinearity: $\mu = (\mu_1, \mu_2) \geq 0$

$$\mathcal{N}(y, p, q; \mu) = \mu_2 \sqrt{y} \sinh(\mu_1 (q - p - \ln y))$$

- Boundary conditions: $y_x(t, a) = y_x(t, b) = p(t, a) = p_x(t, b) = 0$, $q_x(t, a) = q(t, b) = 0$
- Discretization: FE (2nd order) and implicit Euler method
- Numerical solution method: (damped) Newton algorithm

POD basis computation

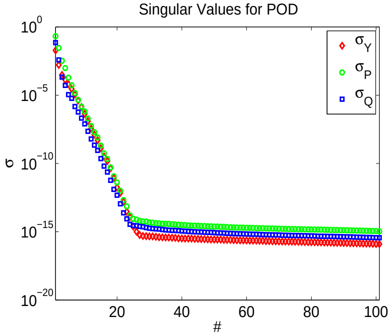
- POD criterium: $\ell \leq \dim(\operatorname{span}\{y(t) \,|\, t \in [0,T]\})$

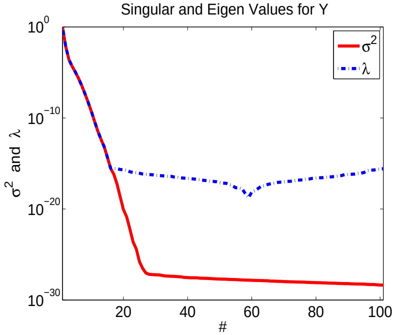
$$\min \int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle \, \psi_i \Big\|^2 \mathrm{d}t \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

- Inner product: $L^2(\Omega)$ or $H^1(\Omega)$ (+b.c.)
- Solution to optimization problem:
  - $\mathcal{R}\psi_i = \int_0^T \langle y(t), \psi_i \rangle \, y(t) \, \mathrm{d}t = \lambda_i \psi_i$, $i = 1, \ldots, \ell$
  - $(\mathcal{K}v_i)(t) = \int_0^T \langle y(t), y(\cdot) \rangle \, v_i \, \mathrm{d}s = \lambda_i v_i(t)$, $i = 1, \ldots, \ell$
  - Relation via SVD: $\psi_i = \int_0^T v_i(t) y(t) \, \mathrm{d}t / \sqrt{\lambda_i}$
- Discrete variant: $\alpha_j = \mathcal{O}(N_t^{-1})$

$$\min \sum_{j=1}^{N_t} \alpha_j \Big\| y(t_j) - \sum_{i=1}^{\ell} \langle y(t_j), \psi_i \rangle \, \psi_i \Big\|^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

Compare: POD and SVD





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- V.'09: Model Reduction using POD. Lecture notes