Proper Orthogonal Decomposition for PDE Constrained Optimization

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General outline of the lecture

1.) The Proper Orthogonal Decomposition (POD) method:

- What is a POD basis?
- How can we compute the basis numerically?
- Which theory is behind?

2.) Reduced-order modelling (ROM) with POD:

- What is a POD reduced-order model?
- How can we derive a-priori error estimates?
- Can we determine an improved ROM in an adaptive way?

3.) POD suboptimal control:

- How do we apply the POD in PDE constrained optimization?
- Can we control the error?
- What can be done for feedback control?

Part 2:

Reduced-Order Modelling (ROM)

- What is a POD reduced-order model?
- How can we derive a-priori error estimates?
- Can we determine an improved ROM in an adaptive way?

Outline of the second part: ROM

- ROM and error estimation
- Applications (λ-ω systems, laser surface hardening, battery)
- Optimal snapshot locations
- References

ROM and error estimation [V.'09]

- Initial value problem in $\mathbb{R}^m$:

$$\dot{y}(t) = Ay(t) + f(t, y(t)) \text{ for } t \in (0, T] \quad \text{and} \quad y(0) = y_0$$

with given $y_0 \in \mathbb{R}^N$ and $f : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$

- Snapshots: $y(t) \in \mathbb{R}^m$ for all $t \in [0, T]$
- POD basis of rank $\ell \leq m$: $\psi_1, \ldots, \psi_\ell \in \mathbb{R}^m$
- Galerkin ansatz: $y^\ell(t) = \sum_{j=1}^{\ell} \left(\psi_j^T y^\ell(t)\right) \psi_j = \sum_{j=1}^{\ell} \mathrm{y}_j^\ell(t)\psi_j$
- Galerkin projection of the ODE:

$$\begin{aligned} \psi_i^T \dot{y}^\ell(t) &= \psi_i^T A y^\ell(t) + \psi_i^T f(t, y^\ell(t)), && t \in (0, T],\ i = 1, \ldots, \ell \\ \psi_i^T y^\ell(0) &= \psi_i^T y_0, && i = 1, \ldots, \ell \end{aligned}$$

POD Galerkin projection of the ODE

- Galerkin projection of the ODE: $f \equiv 0$ (for simplicity)

$$\begin{aligned} \psi_i^T \dot{y}^\ell(t) &= \psi_i^T A y^\ell(t), && t \in (0,T],\ i = 1,\ldots,\ell \\ \psi_i^T y^\ell(0) &= \psi_i^T y_0 && i = 1,\ldots,\ell \end{aligned}$$

- Inserting Galerkin ansatz:

$$\psi_i^T \dot{y}^\ell(t) = \sum_{j=1}^{\ell} \dot{\mathrm{y}}_j^\ell(t) \psi_i^T \psi_j = \dot{\mathrm{y}}_i^\ell(t)$$

$$\psi_i^T A y^\ell(t) = \psi_i^T \left(\sum_{j=1}^{\ell} \mathrm{y}_j^\ell(t) A \psi_j \right) = \sum_{j=1}^{\ell} \mathrm{y}_j^\ell(t) \psi_i^T A \psi_j$$

- ROM in $\mathbb{R}^\ell$: $\mathrm{y}^\ell = (\mathrm{y}_i^\ell)$, $A^\ell = ((\psi_i^T A \psi_j))$, $y_0^\ell = (\psi_i^T y_0)$

$$\begin{aligned} \dot{\mathrm{y}}^\ell(t) &= A^\ell \mathrm{y}(t) && \text{for } t \in (0,T] \\ \mathrm{y}^\ell(0) &= y_0^\ell \end{aligned}$$

Error analysis — Part 1

- Goal: estimate $\int_0^T \|y(t) - y^\ell(t)\|_{\mathbb{R}^m}^2 \,\mathrm{d}t$
- Orthogonal projector onto $V^\ell = \mathrm{span}\,\{\psi_i\}_{i=1}^\ell$:

$$\mathcal{P}^\ell \psi = \sum_{j=1}^{\ell} \left(\psi^T \psi_i\right) \psi_i \quad \text{for } \psi \in \mathbb{R}^m$$

$\Rightarrow y^\ell(0) = \mathcal{P}^\ell y_0 = \mathcal{P}^\ell y(0)$

- POD basis:

$$\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \left(y(t)^T \psi_i\right) \psi_i \Big\|^2 \mathrm{d}t = \int_0^T \Big\| y(t) - \mathcal{P}^\ell y(t) \Big\|^2 \mathrm{d}t$$

- Decomposition:

$$y(t) - y^\ell(t) = \underbrace{y(t) - \mathcal{P}^\ell y(t)}_{\in (V^\ell)^\perp} + \underbrace{\mathcal{P}^\ell y(t) - y^\ell(t)}_{\in V^\ell} = \varrho^\ell(t) + \vartheta^\ell(t)$$

Error analysis — Part 2

- Decomposition:

$$y(t) - y^\ell(t) = y(t) - \mathcal{P}^\ell y(t) + \mathcal{P}^\ell y(t) - y^\ell(t) = \varrho^\ell(t) + \vartheta^\ell(t)$$

- Projector onto $V^\ell = \mathrm{span}\,\{\psi_i\}_{i=1}^\ell$: $\mathcal{P}^\ell \psi = \sum_{j=1}^{\ell} \left(\psi^T \psi_i\right) \psi_i$

- Estimate for ϱ^ℓ:

$$\int_0^T \|\varrho^\ell(t)\|^2 \,\mathrm{d}t = \int_0^T \|y(t) - \mathcal{P}^\ell y(t)\|^2 \,\mathrm{d}t = \sum_{i=\ell+1}^{m} \lambda_i$$

- Differential equation for ϑ^ℓ: for $i \in \{1, \ldots, \ell\}$

$$\begin{aligned}\psi_i^T \dot\vartheta^\ell(t) &= \psi_i^T \left(\mathcal{P}^\ell \dot y(t) - \dot y^\ell(t)\right) = \psi_i^T \left(\dot y(t) - \dot y^\ell(t) + \mathcal{P}^\ell \dot y(t) - \dot y(t)\right)\\ &= \psi_i^T \left(Ay(t) - Ay^\ell(t) + \mathcal{P}^\ell \dot y(t) - \dot y(t)\right)\\ &= \psi_i^T \left(A(\varrho^\ell(t) + \vartheta^\ell(t)) + \mathcal{P}^\ell \dot y(t) - \dot y(t)\right)\end{aligned}$$

Error analysis — Part 3

- Differential equation for ϑ^ℓ: for $i \in \{1, \ldots, \ell\}$

$$\psi_i^T \dot{\vartheta}^\ell(t) = \psi_i^T \left(A(\varrho^\ell(t) + \vartheta^\ell(t)) + \mathcal{P}^\ell \dot{y}(t) - \dot{y}(t) \right)$$

- Summation: $\vartheta^\ell(t) = \sum_{i=1}^\ell c_i(t)\psi_i$

$$\vartheta^\ell(t)^T \dot{\vartheta}^\ell(t) = \vartheta^\ell(t)^T \left(A(\varrho^\ell(t) + \vartheta^\ell(t)) + \mathcal{P}^\ell \dot{y}(t) - \dot{y}(t) \right)$$

- Estimation:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \|\vartheta^\ell(t)\|^2 \le C\Big(\|\vartheta^\ell(t)\|^2 + \|\varrho^\ell(t)\|^2 + \|\dot{y}(t) - \mathcal{P}^\ell \dot{y}(t)\|^2 \Big)$$

- Gronwall's lemma: $\vartheta^\ell(0) = \mathcal{P}^\ell y_0 - y^\ell(0) = 0$

$$\|\vartheta^\ell(t)\|^2 \le C\Big(\|\varrho^\ell(t)\|^2 + \|\dot{y}(t) - \mathcal{P}^\ell \dot{y}(t)\|^2 \Big)$$

$$= C\Big(\sum_{i=\ell+1}^m \lambda_i + \|\dot{y}(t) - \mathcal{P}^\ell \dot{y}(t)\|^2 \Big)$$

Error estimate for continuous POD

- Error estimate (continuous POD method):

$$\int_0^T \|y(t) - y^\ell(t)\|^2 \,\mathrm{d}t \le 2 \int_0^T \|\varrho^\ell(t)\|^2 + \|\vartheta^\ell(t)\|^2 \,\mathrm{d}t$$
$$\le C\Big(\sum_{i=\ell+1}^{m} \lambda_i + \int_0^T \|\dot y(t) - \mathcal{P}^\ell \dot y(t)\|^2 \,\mathrm{d}t \Big)$$

- Remarks:
  - dependence on the decay of the eigenvalues λ_i
  - dependence on the approximation quality for $\dot y(t)$
- Modified POD method:

$$\min \int_0^T \big\|y(t) - \mathcal{P}^\ell y(t)\big\|^2 + \big\|\dot y(t) - \mathcal{P}^\ell \dot y(t)\big\|^2 \,\mathrm{d}t \quad \text{s.t.} \quad \psi_i^T \psi_j = \delta_{ij}$$

- Error estimate: $\displaystyle\int_0^T \|y(t) - y^\ell(t)\|^2 \,\mathrm{d}t \le C \sum_{i=\ell+1}^{m} \tilde\lambda_i$

Extensions [Hömberg/V.'03, Kahlbacher/V.'07, Kunisch/V.'01, Kunisch/V.'02, V.'09]

- Full discrete method: $t_j = j\Delta t,\ Y_j^\ell \approx y(t_j)$

$$\begin{aligned} \psi_i^T \left(\frac{Y_j^\ell - Y_{j-1}^\ell}{\Delta t}\right) &= \psi_i^T A Y_j^\ell + \psi_i^T f(t, Y_j^\ell), \quad && j = 1, \dots, m,\ i = 1, \dots, \ell \\ \psi_i^T Y_0^\ell &= \psi_i^T y_0, && i = 1, \dots, \ell \end{aligned}$$

- Discrete POD: $\lambda_i = \lambda_i^n,\ \psi_i = \psi_i^n$
- Error estimate:

$$\begin{aligned} \sum_{j=1}^n \alpha_j \left\| y(t_j) - Y_j^\ell \right\|_{\mathbb{R}^m}^2 &\le C\left((\Delta t)^2 + \sum_{i=\ell+1}^m \lambda_i^n + \sum_{j=1}^n \alpha_j \left| \dot{y}(t_j)^T \psi_i^n \right|^2 \right) \\ &= O\left((\Delta t)^2 + \sum_{i=\ell+1}^m \left(\lambda_i + \int_0^T \left| \dot{y}(t)^T \psi_i \right|^2 \mathrm{d}t \right) \right) \end{aligned}$$

- nonlinear parabolic PDEs and parameter-dependent elliptic systems

Outline of the second part: ROM

- ROM and error estimation
- **Applications (λ-ω systems, laser surface hardening, battery)**
- Optimal snapshot locations
- References

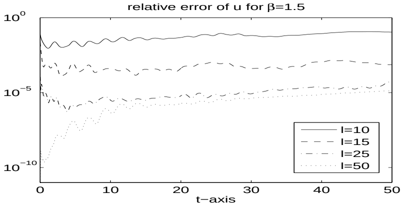
Application 1: λ-ω systems [Müller/V.'06]

- Inner product: $\langle u, v\rangle = \int_\Omega uv \,\mathrm{d}x$
- POD Galerkin ansatz:

$$u_\ell(t,x) = \bar{u}(x) + \sum_{j=1}^{\ell} u_\ell^j(t)\psi_j(x), \quad v_\ell(t,x) = \bar{v}(x) + \sum_{j=1}^{\ell} v_\ell^j(t)\phi_j(x)$$

- Reduced-order model (ROM):
  - insert ansatz into PDEs
  - multiply by POD basis functions ψ_i respectively ϕ_i
  - integrate over Ω
- Numerical results:

$$t \mapsto \frac{\|u_\ell(t) - u(t)\|^2}{\|u(t)\|^2}$$



Application 1: relative POD errors for λ-ω systems

- Offsets: $u_{\mathrm{m}}(x) = \sum_{j=1}^{n} \alpha_j u(t_j, x)$
- Relative POD errors:

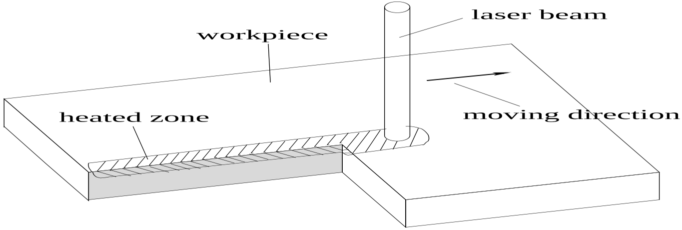
	$\bar u = 0$	$\bar u = u_{\mathrm{m}}$
$\ell = 10$	0.005890	0.005945
$\ell = 15$	0.000350	0.000335
$\ell = 50$	0.000009	0.000009

	$\bar u = 0$	$\bar u = u_{\mathrm{m}}$
$\ell = 40$	0.577442	0.460188
$\ell = 45$	0.898613	0.297619
$\ell = 50$	0.071035	0.001774

$$E_{\mathrm{rel}}(u) = \frac{\sum_{j=1}^{n} \alpha_j \, \|u_\ell(t_j) - u_h(t_j)\|^2}{\sum_{j=1}^{n} \alpha_j \, \|u_h(t_j)\|^2} \quad \text{for } \beta = 1.5 \text{ (left) and } \beta = 2 \text{ (right)}$$

Application 2: laser surface hardening [Hömberg/V.'03]

- Motivation:



- Phase transition of steel:

$$\left.\begin{array}{l}\text{ferrite}\\\text{pearlite}\\\text{bainite}\\\text{martensite}\end{array}\right\}\xrightarrow{\text{heating}}\text{austenite}\xrightarrow{\text{cooling}}\left\{\begin{array}{l}\text{ferrite}\\\text{pearlite}\\\text{bainite}\\\text{martensite}\end{array}\right.$$

Application 2: Model equations for laser surface hardening

- Energy balance and Fourier's law:

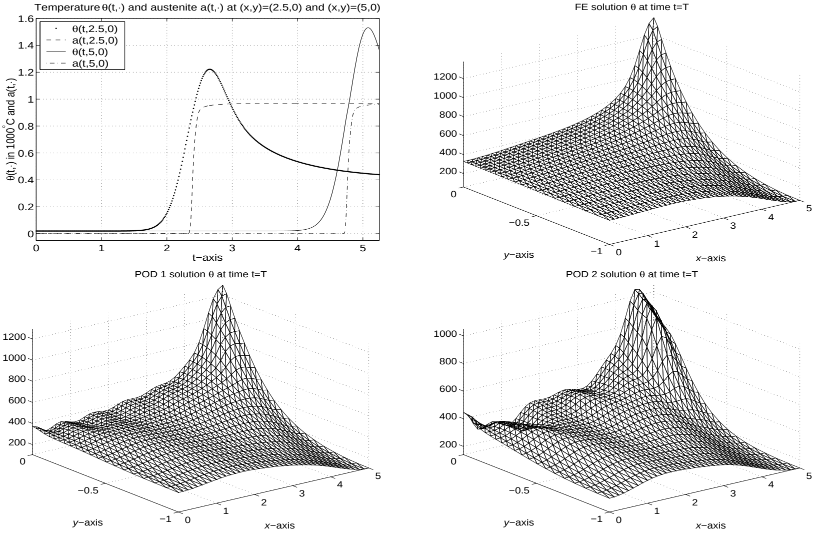
$$\begin{cases} \varrho c_p \theta_t - k\Delta\theta = \alpha u - \varrho L a_t & \text{in } Q = (0,T)\times\Omega \\ \frac{\partial\theta}{\partial n} = 0 & \text{auf } \Sigma = (0,T)\times\partial\Omega \\ \theta(0,\cdot) = \theta_\circ & \text{in } \Omega \subset \mathbb{R}^d \end{cases}$$

- Phase transition of austenite:

$$\begin{cases} a_t = f(\theta,a) & \text{in } Q \\ a(0,\cdot) = 0 & \text{in } \Omega \end{cases}$$

- Intensity of the laser: $u = u(t) \in L^2(0,T)$
- Nonlinearity: $f_+(\theta,a) = \max\{a_{eq}(\theta) - a, 0\}/\tau(\theta)$, $\tau(\theta) > 0$

Application 2: FE and POD temperatures at $t = T$



Application 2: POD error

- Measures for the error:

$$\Psi^i = \frac{\max_{0\le j\le N}\left(\sup_{x\in\Omega}|\theta^j_\ell(x)-\theta^j_{FE}(x)|\right)}{\max_{0\le j\le N}\left(\sup_{x\in\Omega}|\theta^j_{FE}(x)|\right)} \quad \text{with} \quad \begin{cases} i=1 & \text{POD with derivatives} \\ i=2 & \text{POD without derivatives} \end{cases}$$

	$X = L^2(\Omega)$		$X = H^1(\Omega)$	
ℓ	Ψ^1	Ψ^2	Ψ^1	Ψ^2
10	24.1%	40.6%	21.0%	40.1%
25	1.6%	26.9%	4.0%	24.6%

- Heuristic: $\mathcal{E}(\ell) = \sum_{i=1}^{\ell}\lambda_i \Big/ \sum_{i=1}^{d}\lambda_i \cdot 100\% \;\ge 94\%$

	$\ell = 10$	$\ell = 15$	$\ell = 20$	$\ell = 25$
$\mathcal{E}(\ell)$, $X = L^2(\Omega)$	94.3	98.4	99.5	99.8
$\mathcal{E}(\ell)$, $X = H^1(\Omega)$	77.7	87.4	92.5	95.7

Application 3: Elliptic-parabolic systems [Lass/V.'11]

- Elliptic-parabolic systems: $T = 1$, $\Omega = (a, b)$

$$\begin{aligned} y_t - \nabla \cdot (c_1 \nabla y) - \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q = (0, T) \times \Omega \\ -\nabla \cdot (c_2 \nabla p) - \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q \\ -\nabla \cdot (c_3 \nabla q) + \mathcal{N}(y, p, q; \mu) &= 0 && \text{in } Q \end{aligned}$$

- Parameter-dependent nonlinearity: $\mu = (\mu_1, \mu_2) \geq 0$

$$\mathcal{N}(y, p, q; \mu) = \mu_2 \sqrt{y} \sinh(\mu_1(q - p - \ln y))$$

- Boundary conditions: $y_x(t, a) = y_x(t, b) = p(t, a) = p_x(t, b) = 0$, $q_x(t, a) = q(t, b) = 0$

Application 3: ROM with different POD bases for y, p, and q

- Fine model:

$$\begin{aligned} y_t - \nabla\cdot(c_1\nabla y) - \mathcal{N}(y,p,q;\mu) &= 0 \quad \text{in } Q\\ -\nabla\cdot(c_2\nabla p) - \mathcal{N}(y,p,q;\mu) &= 0 \quad \text{in } Q \qquad \text{(FM)}\\ -\nabla\cdot(c_3\nabla q) + \mathcal{N}(y,p,q;\mu) &= 0 \quad \text{in } Q \end{aligned}$$

- FE model for (FM): $y^h(t) = \sum_{i=1}^{N_{FE}} \bar{y}_i(t)\varphi_i$ etc.

$$\begin{aligned} M\bar{y}_t(t) + S_{c_1}\bar{y}(t) - \bar{\mathcal{N}}(\bar{y}(t),\bar{p}(t),\bar{q}(t);\mu) &= 0\\ S_{c_2}\bar{p}(t) - \bar{\mathcal{N}}(\bar{y}(t),\bar{p}(t),\bar{q}(t);\mu) &= 0\\ S_{c_3}\bar{q}(t) + \bar{\mathcal{N}}(\bar{y}(t),\bar{p}(t),\bar{q}(t);\mu) &= 0 \end{aligned}$$

- ROM for (FM): $y^\ell(t) = \sum_{i=1}^{\ell^y} \hat{y}_i(t)\psi_i^y$ etc. [Off-/Online]

$$\begin{aligned} \Psi_y^\top M\Psi_y\hat{y}_t(t) + \Psi_y^\top S_{c_1}\Psi_y\hat{y}(t) - \Psi_y^\top\bar{\mathcal{N}}(y^\ell(t),p^\ell(t),q^\ell(t);\mu) &= 0\\ \Psi_p^\top S_{c_2}\Psi_p\hat{p}(t) - \Psi_p^\top\bar{\mathcal{N}}(y^\ell(t),p^\ell(t),q^\ell(t);\mu) &= 0\\ \Psi_q^\top S_{c_3}\Psi_q\hat{q}(t) + \Psi_q^\top\bar{\mathcal{N}}(y^\ell(t),p^\ell(t),q^\ell(t);\mu) &= 0 \end{aligned}$$

Application 3: Problems for the ROM

- ROM for (FM): $y^\ell(t) = \sum_{i=1}^{\ell y} \hat{y}_i(t)\psi_i^y$ etc.

$$M^{\ell y}\hat{y}_t(t) + S_{c_1}^{\ell y}\hat{y}(t) - \Psi_y^\top \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu) = 0$$
$$S_{c_2}^{\ell p}\Psi_p\hat{p}(t) - \Psi_p^\top \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu) = 0$$
$$S_{c_3}^{\ell q}\hat{q}(t) + \Psi_q^\top \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu) = 0$$

- Problem 1: Imply the reconstruction error

$$\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i\rangle\, \psi_i \Big\|^2 \mathrm{d}t = \sum_{i>\ell} \lambda_i$$

the error relation

$$\|y - y^\ell\|^2 + \|p - p^\ell\|^2 + \|q - q^\ell\|^2 = \mathcal{O}\big(\textstyle\sum_{i>\ell}\lambda_i\big)$$

- Problem 2: evaluation of the nonlinear terms

$$\Psi_y^\top \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu) \text{ etc.}$$

is of complexity $N_{FE} \gg \ell$

Application 3: Problem 1 – A-priori error estimation

- Problem 1: Imply the reconstruction error

$$\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle \, \psi_i \Big\|^2 \, \mathrm{d}t = \sum_{i>\ell} \lambda_i$$

the error relation

$$\|y - y^\ell\|^2 + \|p - p^\ell\|^2 + \|q - q^\ell\|^2 = \mathcal{O}\big(\textstyle\sum_{i>\ell} \lambda_i\big)$$

- Theorem: There is a constant $C > 0$ such that

$$\begin{aligned} \int_0^T & \|y(t) - y^\ell(t)\|^2 + \|p(t) - p^\ell(t)\|^2 + \|q(t) - q^\ell(t)\|^2 \, \mathrm{d}t \\ & \leq C \big(\|\mathcal{P}^{\ell^y} y_\circ - y^\ell(0)\|^2 + \|\mathcal{P}^{\ell^y} y_t - y_t\|^2 \big) \\ & \quad + C \Big(\sum_{i>\ell^y} \lambda_i^y + \sum_{i>\ell^p} \lambda_i^p + \sum_{i>\ell^q} \lambda_i^q \Big) \end{aligned}$$

Application 3: Problem 2 – evaluation of $\Psi_y^\top \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu)$

- Replace: $F(t) = \bar{\mathcal{N}}(y^\ell(t), p^\ell(t), q^\ell(t); \mu) \approx \sum_{i=1}^m c_i(t) u_i \in \mathbb{R}^{N_{FE}}$.
- Interpolation condition for $1 \leq k \leq m \ll N_{FE}$:

$$\big(F(t)\big)_{\mathrm{p}_k} = \Big(\sum_{i=1}^m c(t) u_i\Big)_{\mathrm{p}_k} = \sum_{i=1}^m c_i(t) \big(u_i\big)_{\mathrm{p}_k}, \quad \mathrm{p}_k \in \{1, \ldots, N_{FE}\}$$

- Computation of $c(t)$: $\underbrace{(P^T U)}_{m \times m} c(t) = P^T F(t) \in \mathbb{R}^m$
- Complexity reduction: $P^T F(t) = \bar{\mathcal{N}}(P^T y^\ell(t), P^T p^\ell(t), P^T q^\ell(t); \mu)$
- Choice for U (DEIM): POD basis for $\mathrm{span}\,\{F(t_j)\}_{j=0}^{N_t}$.
- Theorem: error estimate for POD-DEIM compare [Chaturantabut/Sorensen'10]
- Alternative: EIM compare [Grepl, Maday, Ohlberger, Patera, Tonn,...]

Application 3: accuracy (a-priori analysis) for fixed parameter μ

- “Truth” solution: $N_x = 1000$, $N_t = 100$, 2^{nd} order elements
- POD and EIM: $\ell_y = 12$, $\ell_p = 10$, $\ell_q = 10$, $\ell_{DEIM} = \ell_{EIM} = 25$
- Average relative L^2 error (FEM and POD):

	ROM	ROM-EIM	ROM-DEIM
y	1.6765×10^{-7}	1.6763×10^{-7}	1.6762×10^{-7}
p	2.8723×10^{-7}	2.7560×10^{-7}	2.7467×10^{-7}
q	9.7545×10^{-8}	9.4332×10^{-8}	9.1929×10^{-8}

- CPU time:

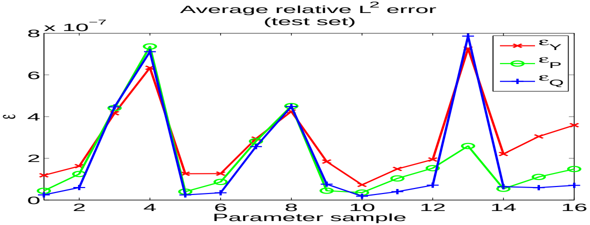
FEM	POD	EIM	DEIM	ROM	ROM-EIM	ROM-DEIM
18.20	0.20	0.19	0.03	6.03	**0.24** ($\approx 1/75$)	**0.48**

Application 3: multiple parameters

- Sample set: $\mu_{sample} \in \{1,2\} \times \{1,2\}$
- Test set: $\mu_{test} \in \{0.5,1.5,2.5,3\} \times \{0.5,1.5,2.5,3\}$
- POD and EIM: $\ell_y = 20$, $\ell_p = 18$, $\ell_q = 18$, $\ell_{EIM} = \ell_{DEIM} = 40$
- CPU time:

FEM	POD	EIM	DEIM	ROM	ROM-EIM	ROM-DEIM
$\sim$ **18**	0.54	0.74	0.09	$\sim$ 7.50	$\sim$ **0.30**	$\sim$ **0.60**

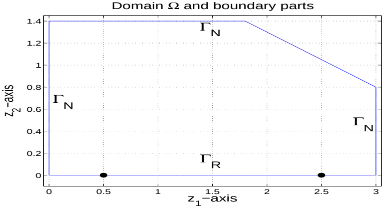
- Average relative L^2 error:

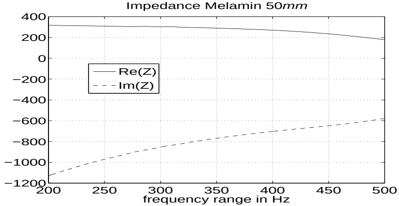


Outline of the second part: ROM

- ROM and error estimation
- Applications (λ-ω systems, laser surface hardening, battery)
- **Optimal snapshot locations**
- References

Acoustic application [Lass/V.'1?]





- Helmholtz equation for frequencies $f \in [f_a, f_b]$:

$$-\Delta p_f - k_f^2\, p_f = q_f \text{ in } \Omega, \qquad \frac{\jmath}{\varrho_\circ\, \omega_f}\,\frac{\partial p_f}{\partial n} = \begin{cases} p_f / Z_f & \text{on } \Gamma_\mathrm{R} \\ 0 & \text{on } \Gamma_\mathrm{N} \end{cases}$$

- Sound pressure $p_f : \Omega \to \mathbb{C}$
- Angular frequency $\omega_f = 2\pi f$, wave number $k_f = \omega_f / c$
- source term $q_f(\mathbf{z}) = c(f) e^{-50\,\|\mathbf{z}-\mathbf{z}_q\|_2^2}$ for $\mathbf{z} = (z_1, z_2) \in \Omega$

Computation of the POD basis (elliptic case)

- Frequency grid: $f_a \le f_1 < \ldots < f_n \le f_b$
- Snapshots: $p_j = p_{f_j}$ for $f = f_j$
- Minimization problem:

$$\min_{\psi_i:\Omega\to\mathbb{C}} \sum_{j=1}^{n} \alpha_j \left\| p_j - \sum_{i=1}^{\ell} \langle p_j, \psi_i \rangle \, \psi_i \right\|^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

- First-order optimality conditions:

$$\mathcal{R}\psi_i = \lambda_i \psi_i, \quad i = 1, \ldots, \ell \qquad (YY^T)$$

with $\mathcal{R}\psi = \sum_{j=1}^{n} \alpha_j \, \langle p_j, \psi \rangle \, p_j : \Omega \to \mathbb{C}$

- Reduced-order model (ROM):

$$\underbrace{\langle -\Delta p_f^\ell - k_f^2 p_f^\ell, \psi_i \rangle}_{\langle \nabla p_f^\ell, \nabla \psi_i \rangle - \langle k_f^2 p_f^\ell, \psi_i \rangle + \text{b.c.}} = \langle q_f, \psi_i \rangle \quad \text{with } p_f^\ell = \sum_{i=1}^{\ell} p_f^i \psi_i$$

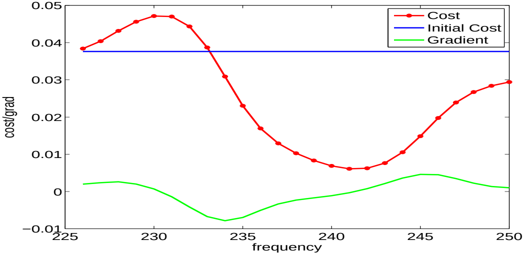
Optimal snapshot locations [Kunisch/V.'10, Lass/V.'1?]

- Given snapshots: p_{f_j} with $f_j = 199 + j$, $j = 1, \ldots, 26$
- Goal: optimal choice of snapshot $p_{\bar f}$ with $\bar f \in [225, 250]$

$$\min_{\bar f \in [225,250]} \frac{1}{2\beta} \int_{200}^{250} \|p_f - p_f^\ell\|^2 \, \mathrm{d}f, \quad \beta = \int_{200}^{250} \|p_f\|^2 \, \mathrm{d}f$$

where p_f^ℓ is computed using the POD basis $\{\psi_i\}_{i=1}^\ell$ with

$$\sum_{j=1}^{26} \alpha_j \langle p_{f_j}, \psi_i \rangle \, p_{f_j} + \underbrace{\bar\alpha \langle p_{\bar f}, \psi_i \rangle \, p_{\bar f}}_{\text{add snapshot } p_{\bar f}} = \lambda_i \psi_i, \quad i = 1, \ldots, \ell$$



Optimal frequency:

$\bar f = 241.35$

Adaptive POD basis computation

Add k frequencies per frequency interval:

1: Choose frequency grid $\mathcal{F}^0 = \{f_j\}_{j=1}^n$, number ℓ of POD functions and number m of frequency intervals;
2: **for** $i = 1, \ldots, m$ **do**
3: Compute optimal frequencies $\{f_j^i\}_{j=1}^k$;
4: Set $\mathcal{F}^i = \mathcal{F}^{i-1} \cup \{f_j^i\}_{j=1}^k$;
5: Enlarge $\ell = \ell + 1$;
6: **end for**

Numerical results

- Frequency grid: $\mathcal{F}^0 = \{199 + j\}_{j=1}^{26} \subset [200, 225]$
- Additional frequency intervals: $m = 5$, more precisely $[225, 250]$, $[250, 275]$, $[275, 300]$, $[300, 325]$, $[325, 350]$
- Comparison of different strategies: $\beta = \int_{200}^{350} \|p_f\|^2 \, \mathrm{d}f$

Snapshot strategy	$\frac{1}{2\beta} \int_{200}^{350} \|p_f - p_f^\ell\|^2 \, \mathrm{d}f$
Add $\{f_1^i\}$	0.0967
Add $\{f_1^i, f_2^i\}$	0.0092
Add $\{224 + j\}_{j=1}^{26}$ etc.	0.0077
Add the mean values	0.1381
No additional snapshots	0.2804

References

- Hömberg/V.'03: Control of laser surface hardening by a reduced-order approach using POD
- Kahlbacher/V.'07: Galerkin POD methods for parameter dependent elliptic systems
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- Kunisch/V.'02: Galerkin POD methods for a general equation in fluid dynamics
- Kunisch/V.'10: Optimal snapshot location for computing POD basis functions
- Lass/V.'11: POD Galerkin schemes for nonlinear elliptic-parabolic systems
- Müller/V.'06: Model reduction by POD for λ-ω systems
- V.'09: Model Reduction using POD. Lecture notes