Proper Orthogonal Decomposition for PDE Constrained Optimization

Stefan Volkwein

Institute of Mathematics and Statistics, University of Constance

Joined work with F. Diwoky, M. Hinze, D. Hömberg, M. Kahlbacher, E. Kammann, K. Kunisch, O. Lass, F. Leibfritz, H. Müller, T. Tonn, F. Tröltzsch, K. Urban

AICES, 3 November 2011

General outline of the lecture

1.) The Proper Orthogonal Decomposition (POD) method:

- What is a POD basis?
- How can we compute the basis numerically?
- Which theory is behind?

2.) Reduced-order modelling (ROM) with POD:

- What is a POD reduced-order model?
- How can we derive a-priori error estimates?
- Can we determine an improved ROM in an adaptive way?

3.) POD suboptimal control:

- How do we apply the POD in PDE constrained optimization?
- Can we control the error?
- What can be done for feedback control?

Part 3:

POD Suboptimal Control

- How do we apply the POD in PDE constrained optimization?
- Can we control the error?
- What can be done for feedback control?

Outline of the third part: POD Suboptimal Control

- Nonlinear heat control
- A-posteriori error analysis
- Optimality-System POD (OS-POD)
- Multilevel SQP
- Static output feedback (SOF)
- References

Nonlinear heat control [Diwoky/V.'01]

- Model problem:

$$\min J(y,u) = \frac{1}{2}\int_\Omega |y(T,x) - z(x)|^2 \,\mathrm{d}x + \frac{\beta}{2}\int_0^T \int_\Gamma |u(t,s)|^2 \,\mathrm{d}s\mathrm{d}t$$

subject to

$$\begin{aligned} y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x) \in Q = (0,T)\times\Omega \\ \frac{\partial y}{\partial n}(t,s) &= b(y(t,s)) + u(t,s) && \text{for } (t,s) \in \Sigma = (0,T)\times\Gamma \\ y(0,x) &= y_\circ(x) && \text{for } x \in \Omega \subset \mathbb{R}^2 \end{aligned}$$

- Assumptions: $T, \beta, k > 0$, $z, y_\circ \in C(\overline{\Omega})$, $b \in C^{2,1}(\mathbb{R})$ with $b' \le 0$

Infinite-dimensional problem

- Optimization variables: $z = (y, u) \in Z$, Z function space
- Equality constraints: $e = (e_1, e_2)$

$$\langle e_1(z), \varphi \rangle = \int_0^T \int_\Omega y_t(t,x)\varphi(t,x) + k\nabla y(t,x) \cdot \nabla\varphi(t,x)\,\mathrm{d}x\mathrm{d}t$$
$$- \int_0^T \int_\Gamma \big(b(y(t,s)) + u(t,s)\big)\varphi(t,s)\,\mathrm{d}s\mathrm{d}t$$
$$e_2(z) = y(0,\cdot) - y_\circ$$

- Infinite-dimensional optimization in function spaces:

$$\min J(z) \quad \text{subject to} \quad e(z) = 0$$

- Lagrange function: $L(z,p) = J(z) + \langle e(z), p \rangle$
- Optimality conditions: $\nabla L(z,p) \stackrel{!}{=} 0$ (Fréchet-derivatives)

First-order optimality conditions

- $\nabla_y L(y,u,p) \overset{!}{=} 0$: adjoint equation

$$\begin{aligned}
-p_t(t,x) &= k\Delta p(t,x) && \text{for } (t,x) \in Q = (0,T)\times\Omega\\
\frac{\partial p}{\partial n}(t,s) &= b'(y(t,s))p(t,s) && \text{for } (t,s) \in \Sigma = (0,T)\times\Gamma\\
p(T,x) &= -(y(T,x) - z(x)) && \text{for } x \in \Omega
\end{aligned}$$

- $\nabla_u L(z,p) \overset{!}{=} 0$: optimality condition $\beta u = kp$ on Σ
- $\nabla_p L(z,p) \overset{!}{=} 0$: state equation

$$\begin{aligned}
y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x) \in Q\\
\frac{\partial y}{\partial n}(t,s) &= b(y(t,s)) + u(t,s) && \text{for } (t,s) \in \Sigma\\
y(0,x) &= y_\circ(x) && \text{for } x \in \Omega
\end{aligned}$$

SQP methods

- SQP: Sequentiel Quadratic Programming
- Quadratic programming problem: $L(z,p) = J(z) + \langle e(z), p\rangle$

$$\begin{aligned} &\min_{z_\delta} L(z^n,p^n) + L_z(z^n,p^n)z_\delta + \tfrac{1}{2} L_{zz}(z^n,p^n)(z_\delta,z_\delta) \\ &\text{subject to } e(z^n) + e'(z^n)z_\delta = 0 \end{aligned} \qquad (\mathrm{QP}^n)$$

- First-order optimality conditions for (QP^n): KKT system

$$\begin{pmatrix} L_{zz}(z^n,p^n) & e'(z^n)^\star \\ e'(z^n) & 0 \end{pmatrix} \begin{pmatrix} z_\delta \\ p_\delta \end{pmatrix} = -\begin{pmatrix} L_z(z^n,p^n) \\ e(z^n) \end{pmatrix}$$

- Convergence: locally quadratic rate in (z^n,p^n) (infinite-dimensional)
- Globalization: modification of the Hessian and line-search methods
- Alternative: trust-region methods

POD model reduction

- Goal: POD Galerkin ansatz using ℓ POD basis functions
- Snapshot POD: solve of heat equation for $0 \le t_1 < \ldots < t_n \le T$
- Problems:
  - unknown optimal control $\Rightarrow$ good snapshot set?
  - $u = \frac{k}{\beta} p$ depends on $p \Rightarrow$ POD approximation for p?
- Strategy: iterate basis computation and include adjoint information in the snapshot ensemble

Dynamic POD strategy [Afanasiev/Hinze'01, Arian/Fahl/Sachs'00, Ravindran'00]

1) Choose estimate u^0; compute snapshots by solving state equation with $u = u^0$ and adjoint equation with $y = y(u^0)$; $i = 0$

2) Determine ℓ POD basis functions and associated ROM of infinite-dimensional optimization problem

3) Compute solution u^{i+1} of optimization problem (e.g., by SQP)

4) If $\Psi(i) = \dfrac{\|u^{i+1} - u^i\|}{\|u^{i+1}\|} \le TOL$ then stop (stopping criterium)

5) $i = i + 1$; compute snapshots by solving state equation with control $u = u^i$ and adjoint equation with $y = y(u^i)$; go back to 2)

Alternative: Optimality-System POD (OS-POD), i.e., change of basis within the optimization w.r.t. optimality conditions

Numerical results

Data: $y_\circ(x_1,x_2)=10x_1x_2$, $z(x_1,x_2)=2+2|2x_1-x_2|$, $b(y)=\arctan(y)$, $k=\beta=\frac{1}{10}$, $T=1$, 185 FEs

Recall: $\Psi(i)=\frac{\|u^{i+1}-u^i\|}{\|u^{i+1}\|}$ stopping criterium for dynamic POD strategy

i	relative L^2 error for y	relative L^2 error for u	$J(y,u)$	$\Psi(i)$
0	4.4	12.0	0.358	1.00
1	1.0	8.1	0.360	0.13
2	0.9	6.8	0.361	0.08
POD_{opt}	0.5	5.7	0.358	
FE			0.358	

		POD	FE
Compute snapshots	M-flops	18	
	CPU time in s	3.3	
Compute POD basis	M-flops	0.44	
	CPU time in s	0.01	
Solve with SQP	M-flops	84	
	CPU time in s	22	
total	M-flops	$1.0\cdot10^2$	$1.9\cdot10^5$
	CPU time in s	$2.5\cdot10^1$	$6.6\cdot10^3$

Outline of the third part: POD Suboptimal Control

- Nonlinear heat control
- **A-posteriori error analysis**
- Optimality-System POD (OS-POD)
- Multilevel SQP
- Static output feedback (SOF)
- References

A-posteriori error analysis [Tröltzsch/V.'09, Kammann/Tröltzsch/V.'11]

- (Abstract) linear-quadratic problem:

$$\min_{(y,u)} \frac{1}{2}\,\|\mathcal{C}y - y_d\|^2 + \frac{\kappa}{2}\,\|u\|^2$$

$$\text{s.t.}\quad \dot{y}(t) = \mathcal{A}(t)y(t) + \mathcal{B}(t)u(t) + f(t),\ \ t \in (0,T], \quad y(0) = y_\circ$$

$$u_a(t) \le u(t) \le u_b(t),\ \ t \in [0,T]$$

- State $y(t) = y(t,\cdot) : \Omega \to \mathbb{R}$ or $\mathbb{C}$
- Input $u(t) = u(t,\cdot)$ (boundary or distributed control)
- $\mathcal{A}$ time-dependent, e.g., $\mathcal{A}(t) = \nabla \cdot (c(t,\cdot)\nabla \bullet) - a(t,\cdot)$
- Control input operator $\mathcal{B}(t)$
- Balanced truncation or moment matching not directly applicable
- Applicable for elliptic problems, extension to nonlinear problems

First-order necessary and sufficient optimality system

- Optimal state $\bar{y}$ and control $\bar{u} \in U_{ad} = \{u \,|\, u_a \le u \le u_b \text{ in } [0,T]\}$
- State equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = \mathcal{A}(t)\bar{y}(t) + \mathcal{B}(t)\bar{u}(t) + f(t),\;\; t \in (0,T], \quad \bar{y}(0) = y_\circ$$

- Adjoint equation:

$$-\frac{\mathrm{d}}{\mathrm{d}t}\bar{p}(t) = \mathcal{A}(t)^\star\bar{p}(t) + \mathcal{C}^\star(y_d - \mathcal{C}\bar{y})(t),\;\; t \in (0,T], \quad \bar{p}(T) = 0$$

  with adjoints $\mathcal{A}(t)^\star$ and $\mathcal{C}^\star$
- Variational inequality:

$$\int_0^T \Big(\kappa\bar{u}(t) - \mathcal{B}(t)^\star\bar{p}(t)\Big)\big(u(t) - \bar{u}(t)\big)\,\mathrm{d}t \ge 0 \quad \forall u \in U_{ad}$$

  with adjoint $\mathcal{B}(t)^\star$

Perturbation analysis

- Arbitrary control: $u^p \in U_{ad} \setminus \{\bar{u}\}$

$$\int_0^T \Big(\kappa u^p(t) - \mathcal{B}(t)^\star p^p(t)\Big)\big(u(t) - u^p(t)\big)\,\mathrm{d}t \not\geq 0 \quad \forall u \in U_{ad}$$

with

$$-\frac{\mathrm{d}}{\mathrm{d}t} p^p(t) = \mathcal{A}(t)^\star p^p(t) + \mathcal{C}^\star(y_d - \mathcal{C}y^p)(t),\;\; t \in (0,T], \quad p^p(T) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t} y^p(t) = \mathcal{A}(t) y^p(t) + \mathcal{B}(t)u^p(t) + f(t),\;\; t \in (0,T], \quad y^p(0) = y_\circ$$

- Perturbation: there exists a computable $\zeta^p = \zeta(u^p)$ [Malanowski, Maurer,...]

$$\int_0^T \Big(\kappa u^p(t) - \mathcal{B}(t)^\star p^p(t) + \zeta^p(t)\Big)\big(u(t) - u^p(t)\big)\,\mathrm{d}t \geq 0 \quad \forall u \in U_{ad}$$

satisfying

$$\boxed{\boxed{\|\bar{u} - u^p\| \leq \frac{1}{\kappa}\,\|\zeta^p\|}}$$

Application for Galerkin-based approximation schemes

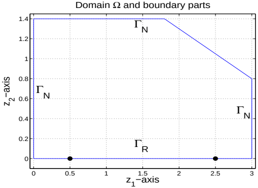
Recall: $\|\bar{u} - u^p\| \leq \frac{1}{\kappa} \|\zeta(u^p)\|$

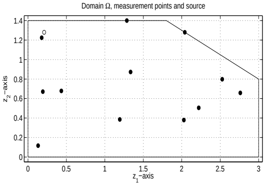
1: Choose basis $\{\psi_i\}_{i=1}^{\ell}$ for Galerkin projection of the control problem;
2: Perform the reduced-order model;
3: Compute suboptimal control $u^p = \bar{u}^\ell$ and perturbation $\bar{\zeta}^\ell = \zeta^\ell(\bar{u}^\ell)$;
4: **if** $\|\bar{\zeta}^\ell\|/\kappa > \mathrm{TOL}$ **then**
5: Enlarge ℓ and go back to step 2;
6: **else**
7: Stop;
8: **end if**

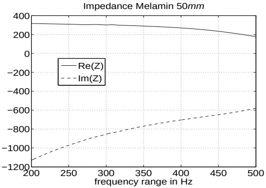
Remarks:

- Applicable for POD and reduced-basis [Tonn/Urban/V.'11]
- $\|\bar{\zeta}^\ell\| \xrightarrow{\ell \to \infty} 0$ for POD [Hinze/V.'08, Tröltzsch/V.'09]
- analogously for elliptic problems [Kahlbacher/V.'07, Kahlbacher/V.'11]
- recent extension to nonlinear problems [Kammann/Tröltzsch/V.'11]

Acoustic example [Tonn/Urban/V.'11]







- Linear-quadratic problem for any frequency $f \in [f_a, f_b]$:

$$\min J_f(y_f, u_f) := \frac{1}{20} \sum_{i=1}^{12} \left| y_f(\mathbf{z}_i) - y_f^i \right|^2 + \frac{1}{2} \left| u_f - u_f^\circ \right|^2$$

$$\text{s.t.} \quad -\Delta y_f - k_f^2\, y_f = u_f\, b \text{ in } \Omega, \quad \frac{\jmath}{\varrho_\circ\, \omega_f}\, \frac{\partial y_f}{\partial n} = \begin{cases} y_f / Z_f & \text{on } \Gamma_{\mathrm{R}} \\ 0 & \text{on } \Gamma_{\mathrm{N}} \end{cases}$$

- State $y_f : \Omega \to \mathbb{C}$ (sound pressure) and control $u_f \in \mathbb{C}$ (intensity)
- $\omega_f = 2\pi f$ and $k_f = \omega_f / c$
- Source term: $b(\mathbf{z}) = e^{-50\, \|\mathbf{z} - \mathbf{z}_q\|_2^2}$ for $\mathbf{z} = (z_1, z_2) \in \Omega$

Computation of the POD basis

- Frequency grid: $f_a \leq f_1 < \ldots < f_n \leq f_b$
- Snapshots: $p_j = p_{f_j} : \Omega \to \mathbb{C}$ solution for "some u", $j = 1, \ldots, n$
- Minimization problem:

$$\min_{\psi_i : \Omega \to \mathbb{C}} \sum_{j=1}^{n} \left\| p_j - \sum_{i=1}^{\ell} \langle p_j, \psi_i \rangle \, \psi_i \right\|^2 \quad \text{s.t.} \quad \langle \psi_i, \psi_j \rangle = \delta_{ij}$$

- Optimality condition:

$$\mathcal{R}\psi_i = \lambda_i \psi_i, \quad i = 1, \ldots, \ell$$

with $\mathcal{R}\psi = \sum_{j=1}^{n} \langle p_j, \psi \rangle \, p_j : \Omega \to \mathbb{C}$

- Reduced-order model (ROM):

$$\underbrace{\langle -\Delta p_f^\ell - k_f^2 p_f^\ell, \psi_i \rangle}_{\langle \nabla p_f^\ell, \nabla \psi_i \rangle - \langle k_f^2 p_f^\ell, \psi_i \rangle + \text{b.c.}} = \langle ub, \psi_i \rangle \quad \text{with } p_f^\ell = \sum_{j=1}^{\ell} p_f^j \psi_j$$

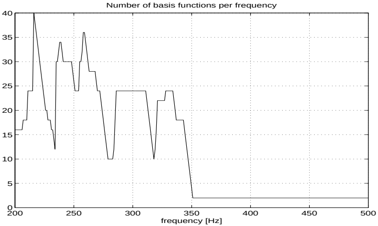
Computation of the reduced-basis [Grepl, Haasdonk, Maday, Ohlberger, Patera, Rozza, Urban, Veroy-Grepl, ...]

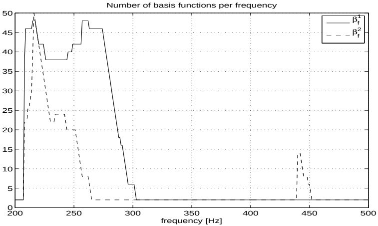
- Frequency grid: $F^\ell = \{f_1, \ldots, f_\ell\} \subset [f_a, f_b]$
- Snapshots: $p_f : \Omega \to \mathbb{C}$, $f \in F^\ell$, solution for "some u"
- Choice of the basis: ONB for $X^\ell = \mathrm{Span}\{p_f : f \in F^\ell\}$
- Relative error estimator: $\Delta_f^\ell \geq \max\limits_{p_f^\ell \in X^\ell} \|p_f - p_f^\ell\| / \|p_f^\ell\|$
- Greedy algorithm:

  1: Choose $f_1 \in [f_a, f_b]$ and set $F^1 = \{f_1\}$, $X^1 = \mathrm{Span}\,\{p_{f_1}\}$, $\ell = 1$;
  2: **while** $\max\limits_{f \in [f_a, f_b]} \Delta_f^\ell > \varepsilon$ **do**
  3: Compute $f^{\ell+1} := \operatorname*{argmax}\limits_{f \in [f_a, f_b]} \Delta_f^\ell$ and set $\ell = \ell + 1$;
  4: Define $F^\ell = F^{\ell-1} \cup \{f^\ell\}$ and $X^\ell = \mathrm{Span}\,\{p_f \,:\, f \in F^\ell\}$;
  5: Compute ONB $\{\psi_i\}_{i=1}^\ell$ for X^ℓ (Gram-Schmidt);
  6: **end while**

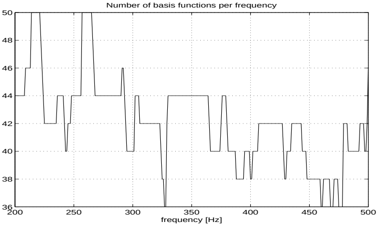
- Reduced-order modell (ROM): analogous to POD

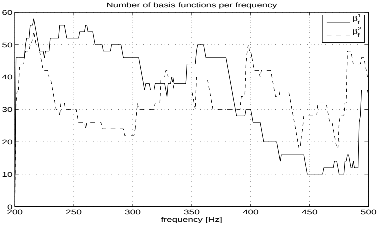
Numerical results for the number $\ell = \ell(f)$ of basis functions





$\|u_f^* - u_f^\ell\| \leq \frac{1}{\kappa}\|\zeta_f^\ell\| \leq 10^{-4}$: $\ell(f)$ for POD (left) & reduced bases (right)





$\|u_f^* - u_f^\ell\| \leq \frac{1}{\kappa}\|\zeta_f^\ell\| \leq 10^{-6}$: $\ell(f)$ for POD (left) & reduced bases (right)

Outline of the third part: POD Suboptimal Control

- Nonlinear heat control
- A-posteriori error analysis
- **Optimality-System POD (OS-POD)**
- Multilevel SQP
- Static output feedback (SOF)
- References

Motivation

- Problem:
  - convergence without any rate in ℓ
  - possibly inappropriate basis for control
- Basis change [Afanasiev/Hinze'01, Arian/Fahl/Sachs'00, Ravindran'00, Willcox et al.'07]:
  - Optimality-System POD (OS-POD) [Kunisch/V.'08]
  - with respect to minimization of the cost
  - combination with a-posteriori analysis [V.'11]

Optimality System POD (OS-POD) [Kunisch/V.'08, Müller'11]

- Original problem:

$$\min J(y,u) \text{ s.t. } \quad \begin{cases} \dot{y}(t) = F(t, y(t), u(t)), & t \in (0,T] \\ y(0) = y_\circ \end{cases}$$

- Approximate problem:

$$\min J(y^\ell,u) \text{ s.t. } \quad \begin{cases} \dot{y}^\ell(t) = F^\ell(t, y^\ell(t), u(t)), & t \in (0,T] \\ y^\ell(0) = y^\ell_\circ \end{cases}$$

Optimality System POD (OS-POD) [Kunisch/V.'08, Müller'11]

- Original problem:

$$\min J(y,u) \text{ s.t. } \quad \begin{cases} \dot{y}(t) = F(t, y(t), u(t)), \;\; t \in (0,T] \\ y(0) = y_\circ \end{cases}$$

- Approximate problem:

$$\min J(y^\ell, u) \text{ s.t. } \quad \begin{cases} \dot{y}^\ell(t) = F^\ell(t, y^\ell(t), u(t)), \;\; t \in (0,T] \\ y^\ell(0) = y^\ell_\circ \end{cases}$$

$$\begin{cases} \int_0^T \langle y(t), \psi_i \rangle \, y(t) \, \mathrm{d}t = \lambda_i \psi_i, \;\; 1 \le i \le \ell \\ \langle \psi_i, \psi_j \rangle = \delta_{ij} \\ \dot{y}(t) = F(t, y(t), u(t)), \;\; t \in (0,T] \\ y(0) = y_\circ \end{cases}$$

- Reduced cost: $\hat{J}_{ospod}(u) = J(y^\ell(u), u, \psi(u), \lambda(u), y(u))$

Combination with a-posteriori analysis [V.'11]

Algorithmus:

1: Choose u^0, compute $y^0 = y^0(u^0)$ and set $j = 0$;
2: Determine POD basis $\{\psi_i\}_{i=1}^{\ell}$ using y^0;
3: Compute m gradient steps for $\hat{J}_{ospod}$ and determine u^m;
4: **if** $\|\zeta^\ell\|/\kappa > \mathrm{TOL}$ **then**
5: Apply a-posteriori algorithm;
6: **else**
7: Stop;
8: **end if**

Remarks:

- Other strategies for the combination possible
- Combine adaptivity (w.r.t. ℓ) and basis changes ($\psi_i = \psi_i(u)$)

Numerical example

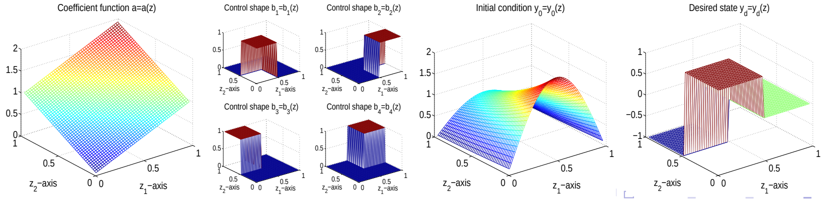
- Linear-quadratic optimal control problem:

$$\min_{(y,u)} \frac{1}{2}\|y-y_d\|^2_{L^2(Q)} + \frac{1}{40}\sum_{i=1}^{4}|u_i|^2$$

$$\text{s.t.}\quad y_t - \Delta y + ay = \sum_{i=1}^{4} u_i b_i + \sin(\pi t) \text{ in } Q=(0,T)\times\Omega,$$

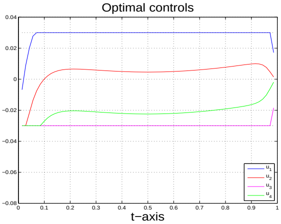
$$y(\cdot\,,0)=0 \text{ auf } \Sigma=(0,T)\times\partial\Omega,\ y(0,\cdot)=y_0 \text{ in } \Omega=(0,1)^2$$

$$-0.03 \le u_i \le 0.03 \text{ in } [0,T],\ 1\le i\le 4$$



Numerical results

- Primal-dual active set strategy for control constraints
- Large-scale optimization: 546 seconds
- POD optimization ($u^0 = 0$, $m = 1$): 15 seconds ($Y^T Y$)
- Estimator: $\|\zeta^\ell\|/\kappa < 6.38\text{e-}6$ for $\ell = 26$
- Error: $\|u^* - u^\ell\| \approx 6.37\text{e-}6 < \frac{1}{2}\max(h^2, \Delta t^2) \approx 2\text{e-}4 =: \mathrm{TOL}$



- A-posteriori algorithm without OS-POD:

$$\|\zeta^\ell\|/\kappa \approx 1.23\text{e-}3 \not\leq \mathrm{TOL} < 6.38\text{e-}6$$

for $\ell = 50(!)$

Outline of the third part: POD Suboptimal Control

- Nonlinear heat control
- A-posteriori error analysis
- Optimality-System POD (OS-POD)
- **Multilevel SQP**
- Static output feedback (SOF)
- References

Multilevel SQP [Kahlbacher/V.'10, Sachs/V.'10]

- Infinite dimensional optimization:

$$\min J(x) \quad \text{s.t.} \quad e(x) = 0 \qquad \textbf{(P)}$$

- Lagrange functional for **(P)**: $\mathcal{L}(x,p) = J(x) + \langle e(x), p \rangle$
- (Local) SQP method: at $z_k = (x_k, p_k)$ solve

$$\begin{cases} \min\limits_{x_\delta} \mathcal{L}(x_k, p_k) + \mathcal{L}_x(z_k)x_\delta + \frac{1}{2}\mathcal{L}_{xx}(z_k)(x_\delta, x_\delta) \\ \text{subject to } e(x_k) + e'(x_k)x_\delta = 0 \end{cases} \qquad (\mathbf{QP}^k)$$

- KKT system: solution $\bar{x}_\delta$ to $(\mathbf{QP}^k)$ is characterized by

$$\underbrace{\begin{pmatrix} \mathcal{L}_{xx}(z_k) & e'(x_k)^\star \\ e'(x_k) & 0 \end{pmatrix}}_{A_k} \cdot \underbrace{\begin{pmatrix} \bar{x}_\delta \\ \bar{p}_\delta \end{pmatrix}}_{\bar{z}_\delta} = \underbrace{-\begin{pmatrix} \mathcal{L}_x(z_k) \\ e(x_k) \end{pmatrix}}_{b_k}$$

Inexact SQP by using POD or RB

- KKT system: inexact solve of $A_k \bar{z}_\delta = b_k$ by discretization
- Discretization: (POD or RB or BT or...) model reduction

$$A_k^\ell \bar{z}_\delta^\ell = b_k^\ell \in \mathbb{R}^n, \quad n = n(\ell)$$

- Convergence of (local) SQP method: $\bar{z}_\delta^\ell$ reduced-order solution

$$\|A_k \mathcal{P} \bar{z}_\delta^\ell - b_k\| = \mathcal{O}\big(\|\mathcal{L}'(z_k)\|^q\big), \quad q \in [1,2]$$

  with prolongation $\mathcal{P}$
- Rate of convergence: superlinear $(1 < q < 2)$, quadratic $(q = 2)$
- Control of reduced-order approach:

$$\|A_k \mathcal{P} \bar{z}_\delta^\ell - b_k\| \simeq \|\bar{z}_\delta - \mathcal{P} \bar{z}_\delta^\ell\| \simeq \|\mathcal{L}'(z_k)\|^q$$

Local convergence result

- Variables in optimal control: $x = (y, u)$, $y = y(u)$
- KKT system: $z_k = (x_k, p_k)$, $x_k = (y_k, u_k)$

$$\left(\begin{array}{cc|c} \mathcal{L}_{yy}(z_k) & \mathcal{L}_{yu}(z_k) & e_y(x_k)^\star \\ \mathcal{L}_{uy}(z_k) & \mathcal{L}_{uu}(z_k) & e_u(x_k)^\star \\ \hline e_y(x_k) & e_u(x_k) & 0 \end{array}\right) \begin{pmatrix} y_\delta \\ u_\delta \\ p_\delta \end{pmatrix} = \begin{pmatrix} -\mathcal{L}_y(z_k) \\ -\mathcal{L}_u(z_k) \\ -e(x_k) \end{pmatrix}$$

- Suboptimal solution to KKT system: $\bar{z}_\delta^\ell = (\bar{y}_\delta^\ell, \bar{u}_\delta^\ell, \bar{p}_\delta^\ell)$
- Prolongation $\mathcal{P}$: $\bar{z}_\delta^\ell \mapsto \mathcal{P}\bar{z}_\delta^\ell = (\tilde{y}_\delta, \bar{u}_\delta^\ell, \tilde{p}_\delta)$ with

$$e_y(x_k)\tilde{y}_\delta = -e(x_k) - e_u(x_k)\bar{u}_\delta^\ell$$
$$e_y(x_k)^\star\tilde{p}_\delta = -\mathcal{L}_y(z_k) - \mathcal{L}_{yy}(z_k)\tilde{y} - \mathcal{L}_{yu}(z_k)\bar{u}_\delta^\ell$$

- Theorem: second-order sufficient optimality implies

$$\lim_{k\to\infty} z_k + \mathcal{P}\bar{z}_\delta^\ell = \bar{z} \quad \text{if} \quad \|A_k\mathcal{P}\bar{z}_\delta^\ell - b_k\| \simeq \|\bar{u}_\delta - \bar{u}_\delta^\ell\| < \mathrm{TOL}$$

Multilevel approach with reduced-order models

- Convergence criterium: $\|A_k \mathcal{P} \bar{z}_\delta^\ell - b_k\| \simeq \|\bar{u}_\delta - \bar{u}_\delta^\ell\| < \mathrm{TOL}$
- A-posteriori error [Tröltzsch/V.'09]:

$$\|\bar{u}_\delta - \bar{u}_\delta^\ell\| \simeq \| \underbrace{\mathcal{L}_{uy}(z_k)\tilde{y}_\delta + \mathcal{L}_{uu}(z_k)\bar{u}_\delta^\ell + e_u(x_k)^\star \tilde{p}_\delta + \mathcal{L}_u(z_k)}_{:=-\bar{\zeta}^\ell} \|$$

  with $\|\bar{\zeta}^\ell\| \to 0$ for $\ell \to \infty$
- Convergence of $\|\bar{\zeta}^\ell\|$: no rate, basis dependent [Hinze/V.'08]
- POD basis: combination with Optimality-System POD [V.'11]
- Alternatives via nonlinear optimization: Trust-Region POD [Arian/Fahl/Sachs'00, Schu/Sachs'07]
- Combination with adaptive schemes [Clever/Lang/Ulbrich/Ziems]

Numerical experiments

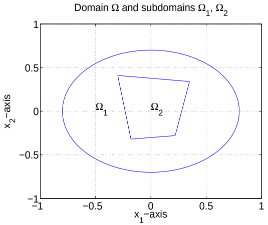
- Optimal control problem:

$$\min \frac{1}{2}\int_{\Omega_m} |y - y^d|^2 \,\mathrm{d}x + \frac{\kappa}{2}\sum_{i=1}^{n_\Omega} |\Omega_i||u_i - u_i^\circ|^2$$

$$\text{s.t. } u_i \geq 0, \quad -\Delta y + y\sum_{i=1}^{n_\Omega} u_i\chi_{\Omega_i} = 0 \text{ in } \Omega, \ \frac{\partial y}{\partial n} + f = g \text{ on } \Gamma$$

- Given data: y^d, u_i°, $\kappa > 0$, f, g
- Globalization of SQP:
    - modification of the hessian
    - Armijo linesearch with ℓ_1 merit function
- Equality constraint case: $x^* = (y^*, u^*)$ with inactive $u^* > 0$

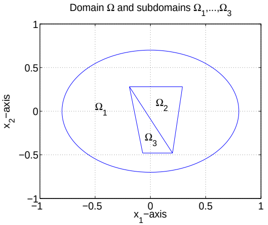
Numerical examples



Reference control: u°

Desired state: $y(u^\circ)+$ noise

Measurement domain: $\Omega_m \subsetneq \Omega$



SQP it.	$\|\mathcal{L}'(z_{k-1})\|$	a-post	ℓ
$k=1$	1.38e-0	1.16e−3	6
$k=2$	8.53e-1	8.36e−2	12
$k=3$	2.57e-1	7.87e−5	12
$k=4$	4.65e-3	5.67e−6	12
$k=4$	3.05e-5	1.90e−6	12
$k=6$	4.66e-9	—	—

SQP it.	$\|\mathcal{L}'(z_k)\|$	a-post	ℓ
$k=1$	3.01e-2	1.58e-3	10
$k=2$	4.48e-1	7.44e-4	10
$k=3$	3.63e-1	3.46e-4	10
$k=4$	5.16e-2	4.04e-5	10
$k=5$	1.48e-2	4.98e-4	10
$k=6$	1.62e-3	6.56e-4	10
$k=7$	7.51e-7	—	—

Outline of the third part: POD suboptimal control

- Nonlinear heat control
- A-posteriori error analysis
- Acoustic example
- Multilevel SQP
- **Static output feedback (SOF)**
- References

Linear-quadratic-regulator (LQR) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for } t > 0, \quad x(0) = x_\circ$$

  with state $x(t) \in \mathbb{R}^\ell$, control $u(t) \in \mathbb{R}^{n_u}$ and $A \in \mathbb{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$

- Quadratic cost: $J(x,u) = \displaystyle\int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)\,\mathrm{d}t$

  with $Q \in \mathbb{R}^{\ell\times\ell}$, $Q \succeq 0$ and $R \in \mathbb{R}^{n_u\times n_u}$, $R \succ 0$

- Goal: (full state) feedback law $u(t) = Fx(t)$ with $F \in \mathbb{R}^{n_u\times\ell}$
- Solution: $F = -R^{-1}B^TP$ with $P = P^T \in \mathbb{R}^{\ell\times\ell}$

$$A^TP + PA + Q - PBR^{-1}B^TP = 0 \qquad \text{(Matrix Riccati)}$$

- Problem: often only partial state measurement available

$\mathcal{H}_2$ static output feedback (SOF) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ$$
$$y(t) = Cx(t)$$

with $A \in \mathrm{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$, $B_1 \in \mathbb{R}^{\ell\times n_w}$, $C \in \mathbb{R}^{n_y\times\ell}$ and

$$x(t) \in \mathbb{R}^\ell, \quad u(t) \in \mathbb{R}^{n_u}, \quad y(t) \in \mathbb{R}^{n_y}, \quad w(t) \in \mathbb{R}^{n_w}$$

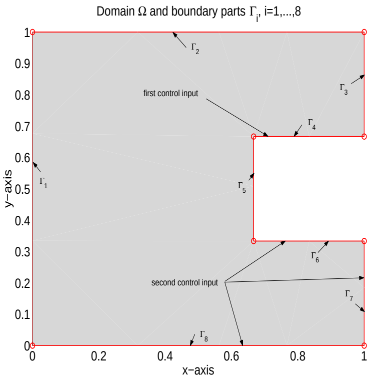
- Feedback law: $u(t) = Fy(t)$ with $F \in \mathbb{R}^{n_u\times n_y}$
- Solution: F given by nonconvex semidefinite programming

$$\min \operatorname{trace}\left(LB_1B_1^T\right) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \in \mathbb{R}^{\ell\times\ell} \quad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2\ell\times\ell}$

SOF controller design [Leibfritz/V.'06]

$$
\begin{aligned}
v_t &= \kappa \Delta v + a v && \text{in } \Omega \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (0,T),\ j=1,2,3,5 \\
-\lambda \frac{\partial v}{\partial n} &= \alpha_4 (v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= \hat{\alpha}(v - \hat{c} + \hat{u}(t)) && \text{on } \Gamma_j \times (0,T),\ j=6,7,8 \\
v(0) &= v_\circ && \text{in } \Omega
\end{aligned}
$$



Control: $u(t) \in \mathbb{R}^2,\ n_u = 2$

Measurement: $y(t) \in \mathbb{R}^3,\ n_y = 3$

$$
\begin{aligned}
y_1(t) &= v(0,1;t) \\
y_2(t) &= v(0,0;t) \\
y_3(t) &= v(2/3,1/2;t)
\end{aligned}
$$

Goal: $u(t) = Fy(t),\ F \in \mathbb{R}^{2\times 3}$

Variational form for nonlinear heat equation

- Nonlinear heat equation:

$$
\begin{aligned}
v_t &= \kappa\Delta v + av && \text{in } \Omega\times(0,T)\\
-\lambda\frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j\times(0,T),\ j=1,2,3,5\\
-\lambda\frac{\partial v}{\partial n} &= \alpha_4(v-c_4+u_4(t))+\varepsilon_4\sigma(v^4-c_4^4) && \text{on } \Gamma_4\times(0,T)\\
-\lambda\frac{\partial v}{\partial n} &= \hat\alpha(v-\hat c+\hat u(t)) && \text{on } \Gamma_j\times(0,T),\ j=6,7,8
\end{aligned}
$$

- Variational form: for all $\varphi\in H^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega v_t(t)\varphi+\kappa\nabla v(t)\cdot\nabla\varphi-av(t)\varphi\,\mathrm{d}x=\kappa\int_\Gamma\frac{\partial v(t)}{\partial n}\varphi\,\mathrm{d}s=\frac{\kappa}{\lambda}\int_\Gamma\lambda\frac{\partial v(t)}{\partial n}\varphi\,\mathrm{d}s\\
&=\frac{\kappa}{\lambda}\int_{\Gamma_4}(\alpha_4c_4+\varepsilon_4\sigma c_4^4)\varphi-(\alpha_4v(t)+\varepsilon_4\sigma v^4(t))\varphi-\alpha_4u_4(t)\varphi\,\mathrm{d}s\\
&\quad+\frac{\kappa}{\lambda}\int_{\Gamma_6\cup\Gamma_7\cup\Gamma_8}\hat\alpha\hat c\varphi-\hat\alpha v(t)\varphi-\hat\alpha\hat u(t)\varphi\,\mathrm{d}s
\end{aligned}
$$

$\mathcal{H}_2$ SOF design

- Dynamical system in $\mathbb{R}^N$: spatial discretization (e.g., FE or FD) and linearization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ \\ y(t) &= Cx(t) \end{aligned}$$

- Goal: feedback law $u(t) = Fy(t)$ with $F \in \mathbb{R}^{2\times 3}$
- Solution: F given by

$$\min \operatorname{trace}\left(LB_1 B_1^T\right) \quad \text{s.t.} \quad H(F, L, V) = 0 \;\&\; V \succ 0 \qquad \text{(SDP)}$$

with $H(F, L, V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2N\times N}$

- $N = \#$ FE or FD unknowns (!)

Reduced-order model (ROM)

- Compute solution y of nonlinear heat quation with FE or FD at time instances $0 \leq t_1 < \ldots < t_n \leq T$
- Snapshots: $y_j = y(t_j)$ for $i = 1, \ldots, n$
- POD: $\mathcal{R}^n \psi_i = \lambda_i \psi_i$ with $\mathcal{R}^n \psi_i = \sum_{j=1}^{n} \alpha_j \int_\Omega \psi_i y_j \,\mathrm{d}x \; y_j$
- ROM: Galerkin ansatz for nonlinear heat equation with $\psi_1, \ldots, \psi_\ell$

$$\begin{aligned} \dot{x}(t) &= A^\ell x(t) + G^\ell(x(t)) + B^\ell u(t) + B_1^\ell w(t), \quad x(0) = x_\circ^\ell \\ y(t) &= C^\ell x(t) \\ u(t) &= F^\ell y(t), \quad F^\ell \in \mathbb{R}^{2\times 3} \end{aligned}$$

Feedback synthesis

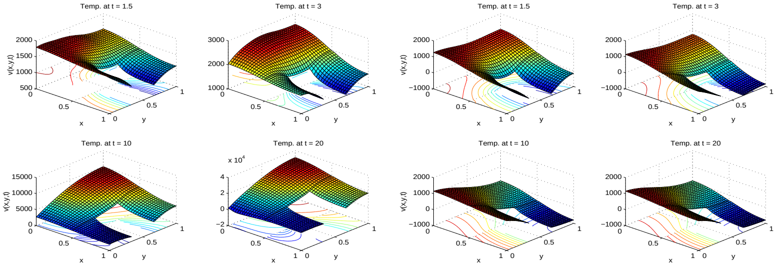
- Reduction in the variable x, not in y and u
- Linearize and set up the SDP problem
  $\Rightarrow$ ℓ is the size of the SDP problem
  $\Rightarrow$ $5 = \ell \ll 3796$ FD unknowns
- Solve SDP by Interior-point trust-region method [Leibfritz/Mostafa]
- Plug in the computed feedback law into the FD modell (closed-loop)

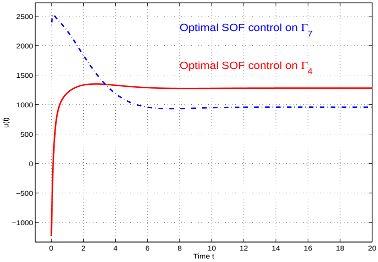
$$\dot{x}(t) = Ax(t) + G(x(t)) + B \underbrace{F^\ell Cx(t)}_{=F^\ell y(t)=u(t)} + B_1 w(t), \; x(0) = x_\circ$$

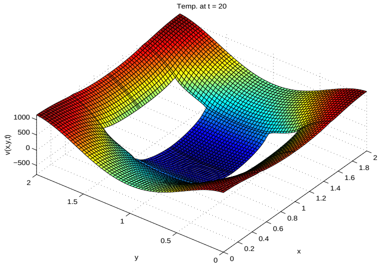
$$y(t) = Cx(t)$$

$$u(t) = F^\ell y(t) = F^\ell Cx(t)$$

Numerical example (Part 3)







References

- Diwoky/V.'01: Nonlinear boundary control for the heat equation utilizing POD
- Hinze/V.'08: Error estimates for abstract linear-quadratic optimal control problems using POD
- Kunisch/V.'08: POD for optimality systems
- Leibfritz/V.'04: Reduced order output feedback control design for PDE systems using POD and nonlinear semidefinite programming
- Sachs/V.'10: POD-Galerkin approximations in PDE-constrained optimization
- Tonn/Urban/V.'11: Comparison of the reduced-basis and POD a-posteriori error estimators for an elliptic linear-quadratic optimal control problem
- Tröltzsch/V.'09: POD a-posteriori error estimates for linear- quadratic optimal control problems
- V.'11: POD a-posteriori error estimates for linear-quadratic optimal control problems
- Kammann/Tröltzsch/V.'11: A method of a-posteriori error estimation with application to POD

Ongoing research

- A-posteriori analysis for nonlinear problems [Kammann/Tröltzsch/V., Trenz/V.]
- OS-POD for mixed control-state constraints [Gubisch/V.]
- POD for battery equations [Lass/V.]
- Stable KKT approximations [Gerner/Veroy-Grepl/V.]
- Comparison of various model-order strategies [Vossen/V.]
- A-posteriori error analysis for reduced basis schemes [Grepl/Kärcher/V.]