POD in Feedback Strategies

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Open and Closed Loop Control

- **Open loop control**:

$$\text{input } u(t) \to \boxed{\text{plant}} \to \text{ state } x(t) \in \mathbb{R}^\ell$$

- **Closed loop control**: determine mapping $\mathscr{F}$ such that

$$u(t) = \mathscr{F}(t, x(t)) \qquad \text{(feedback law)}$$

- **Linear-quadratic case**: LQR and LQG design
- **Nonlinear case**: Hamilton-Jacobi-Bellman (HJB) equations

$$v_t(t,x) + H(\nabla_x v(t,x), x) = 0 \quad \text{for } (t,x) \in (t_\circ, t_f) \times \mathbb{R}^\ell$$

Outline of Lecture 3

- **Static Output Feedback Synthesis**
- **Hamilton-Jacobi-Bellman (HJB) Based Feedback Design**
- **POD based Model Predictive Control**

Static Output Feedback (SOF) Synthesis (Leibfritz/V.'06)

Linear-Quadratic Regulator (LQR) Design

- **Linear dynamical system** in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for } t > 0, \quad x(t_\circ) = x_\circ$$

  with state $x(t) \in \mathbb{R}^\ell$, control $u(t) \in \mathbb{R}^{n_u}$ and $A \in \mathbb{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$

- **Cost**: $J(x,u) = \int_{t_\circ}^\infty x(t)^\top Qx(t) + u(t)^\top Ru(t)\,\mathrm{d}t$

  with $Q \in \mathbb{R}^{\ell\times\ell}$, $Q \succeq 0$ and $R \in \mathbb{R}^{n_u\times n_u}$, $R \succ 0$

- **Goal**: (full state) feedback law $u(t) = Fx(t)$ with $F \in \mathbb{R}^{n_u\times\ell}$

- **Solution**: $F = -R^{-1}B^\top P$ with $P = P^\top \in \mathrm{R}^{\ell\times\ell}$

$$A^\top P + PA + Q - PBR^{-1}B^\top P = 0 \qquad \text{(Matrix Riccati)}$$

- **Problem**: often only partial state measurement available

$\mathcal{H}_2$ Static Output Feedback Design

- **Linear dynamical system** in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \quad x(t_\circ) = x_\circ$$
$$y(t) = Cx(t)$$

with $A \in \mathrm{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$, $B_1 \in \mathbb{R}^{\ell\times n_w}$, $C \in \mathbb{R}^{n_y\times\ell}$ and

$$x(t) \in \mathbb{R}^\ell, \quad u(t) \in \mathbb{R}^{n_u}, \quad y(t) \in \mathbb{R}^{n_y}, \quad w(t) \in \mathbb{R}^{n_w}$$

- **Feedback law**: $u(t) = Fy(t)$ with $F \in \mathbb{R}^{n_u\times n_y}$
- **Solution:** F given by nonconvex semidefinite programming

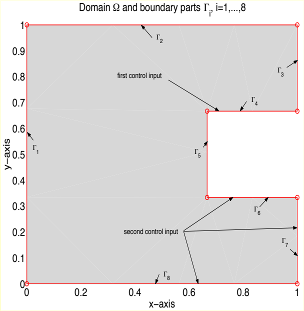
$$\min \operatorname{trace}(LB_1B_1^\top) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \in \mathbb{R}^{\ell\times\ell} \tag{SDP}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^\top L + LA(F) + C(F)^\top C(F) \\ A(F)^\top V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2\ell\times\ell}$

Static Output Feedback Controller Design (Leibfritz/V.'06)

Nonlinear heat equation:

$$
\begin{aligned}
v_t &= \kappa \Delta v + a v && \text{in } \Omega \times (t_\circ, t_f) \\
-\lambda \frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (t_\circ, t_f),\ j = 1,2,3,5 \\
-\lambda \frac{\partial v}{\partial n} &= \alpha_4 (v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (t_\circ, t_f) \\
-\lambda \frac{\partial v}{\partial n} &= \hat{\alpha}(v - \hat{c} + \hat{u}(t)) && \text{on } \Gamma_j \times (t_\circ, t_f),\ j = 6,7,8 \\
v(t_\circ) &= v_\circ && \text{in } \Omega
\end{aligned}
$$



Control: $u(t) \in \mathbb{R}^2$, $n_u = 2$

Measurement: $y(t) \in \mathbb{R}^3$, $n_y = 3$

$$
\begin{aligned}
y_1(t) &= v(0,1;t) \\
y_2(t) &= v(0,0;t) \\
y_3(t) &= v(2/3,1/2;t)
\end{aligned}
$$

Goal: $u(t) = Fy(t)$, $F \in \mathbb{R}^{2\times 3}$

Variational Form for the Nonlinear Heat Equation

- **Nonlinear heat equation**:

$$
\begin{aligned}
v_t &= \kappa \Delta v + a v && \text{in } \Omega \times (t_\circ, t_f)\\
-\lambda \tfrac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (t_\circ, t_f),\ j = 1,2,3,5\\
-\lambda \tfrac{\partial v}{\partial n} &= \alpha_4 (v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (t_\circ, t_f)\\
-\lambda \tfrac{\partial v}{\partial n} &= \hat{\alpha}(v - \hat{c} + \hat{u}(t)) && \text{on } \Gamma_j \times (t_\circ, t_f),\ j = 6,7,8
\end{aligned}
$$

- **Variational form**: for all $\varphi \in H^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega v_t(t)\varphi + \kappa \nabla v(t) \cdot \nabla \varphi - a v(t) \varphi \,\mathrm{d}x = \kappa \int_\Gamma \frac{\partial v(t)}{\partial n} \varphi \,\mathrm{d}s = \frac{\kappa}{\lambda} \int_\Gamma \lambda \frac{\partial v(t)}{\partial n} \varphi \,\mathrm{d}s\\
&= \frac{\kappa}{\lambda} \int_{\Gamma_4} (\alpha_4 c_4 + \varepsilon_4 \sigma c_4^4)\varphi - (\alpha_4 v(t) + \varepsilon_4 \sigma v^4(t))\varphi - \alpha_4 u_4(t) \varphi \,\mathrm{d}s\\
&\quad + \frac{\kappa}{\lambda} \int_{\Gamma_6 \cup \Gamma_7 \cup \Gamma_8} \hat{\alpha}\hat{c}\varphi - \hat{\alpha} v(t) \varphi - \hat{\alpha}\hat{u}(t)\varphi \,\mathrm{d}s
\end{aligned}
$$

$\mathscr{H}_2$ Static Output Feedback Design

- **Dynamical system** in $\mathbb{R}^N$: spatial discretization (e.g., FE or FD) and linearization

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(t_\circ) = x_\circ$$
$$y(t) = Cx(t)$$

- **Goal**: feedback law $u(t) = Fy(t)$ with $F \in \mathbb{R}^{2\times 3}$
- **Solution**: F given by

$$\min \operatorname{trace}(LB_1 B_1^\top) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \qquad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^\top L + LA(F) + C(F)^\top C(F) \\ A(F)^\top V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2N\times N}$

- **Problem**: $N = \#$ FE or FD unknowns (!)

Reduced-Order Modeling with POD

- **Snapshots generation**: Compute solution y of nonlinear heat equation with FE or FD at time instances $t_\circ \leq t_1 < \ldots < t_n \leq t_f$
- **Snapshots**: $y_j = y(t_j) \in H = L^2(\Omega)$ for $i = 1, \ldots, n$
- **POD**: $\mathscr{R}^n \psi_i = \lambda_i \psi_i$ with $\mathscr{R} = \sum_{j=1}^{n} \alpha_j \langle \cdot, y_j \rangle_H y_j$
- **Reduced-Order Modeling**: Galerkin ansatz for nonlinear heat equation with $\psi_1, \ldots, \psi_\ell$

$$\begin{aligned}
\dot{x}(t) &= A^\ell x(t) + G^\ell(x(t)) + B^\ell u(t) + B_1^\ell w(t), \quad x(t_\circ) = x_\circ^\ell \\
y(t) &= C^\ell x(t) \\
u(t) &= F^\ell y(t), \quad F^\ell \in \mathbb{R}^{2\times 3}
\end{aligned}$$

Feedback Synthesis

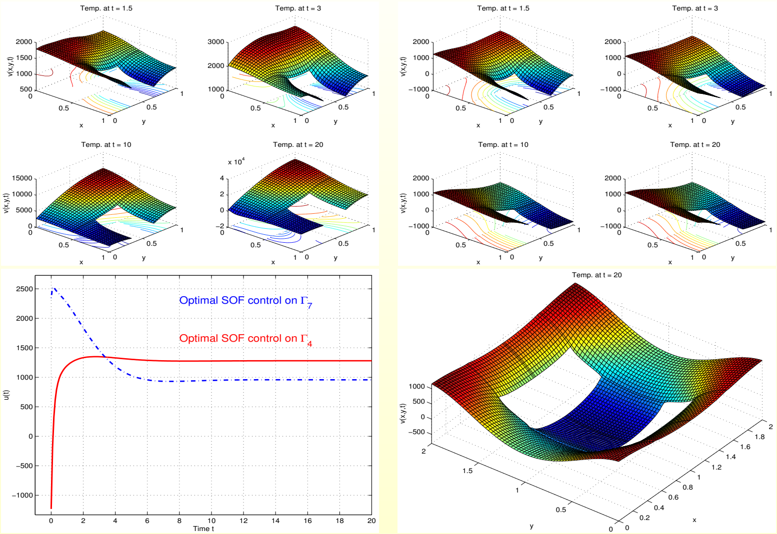
- Reduction in the variable x, not in y and u
- Linearize and set up the SDP problem
  $\Rightarrow \ell$ is the size of the SDP problem
  $\Rightarrow 5 = \ell \ll 3796$ FD unknowns
- Solve SDP by **Interior-point trust-region method** (Leibfritz/Mostafa)
- Plug in the computed feedback law into the FD modell (**closed-loop**)

$$\dot{x}(t) = Ax(t) + G(x(t)) + B \underbrace{F^\ell C x(t)}_{=F^\ell y(t) = u(t)} + B_1 w(t), \; x(t_\circ) = x_\circ$$

$$y(t) = Cx(t)$$

$$u(t) = F^\ell y(t) = F^\ell C x(t)$$

Numerical Example



HJB Based Feedback Design (Kunisch/V./Xie'04)

Hamilton-Jacobi-Bellman Equation

- **Dynamical system** in $\mathbb{R}^\ell$:

$$\begin{cases} \dot y(t) = F(y(t),u(t)) & \text{for } t > t_\circ \\ y(t_\circ) = y_\circ \end{cases} \tag{1}$$

- **Admissible controls**: $u \in L^2(t_\circ, t_f; \mathbb{R}^m)$, $u(t) \in U \subset \mathbb{R}^m$
- **Cost**:

$$\min J(u;t_\circ,y_\circ) = \int_{t_\circ}^\infty L(y(t),u(t))e^{-\mu t}\,\mathrm{d}t$$

with $y = y(u)$ solution to (1) and $\mu > 0$

- **Euler's method**: $y_{j+1} = y_j + hF(y_j,u_j)$ for $j \geq 0$
- **Discrete cost**:

$$J_h(u;t_\circ,y_\circ) = \frac{h}{2}\Big(L(y_\circ,u_0) + \sum_{j=1}^\infty e^{-\mu jh}\big[L(y_j,u_{j-1}) + L(y_j,u_j)\big]\Big)$$

- **Discrete minimal value function**:

$$v_h(y_\circ) = \inf\{J_h(u_h;t_\circ,y_\circ) : u_h \in \mathscr{U}^h\}$$

with $\mathscr{U}^h = \{u_h = \{u_0,u_1,\ldots\} \mid u_j \in U\}$

- **Discrete HJB equation**: for all $y_\circ \in \mathbb{R}^\ell$ and $\beta = e^{-\mu h}$

$$v_h(y_\circ) = \inf_{u\in U}\Big\{\frac{h}{2}[L(y_\circ,u) + \beta L(y_\circ + hF(y_\circ,u),u)] + \beta v_h(y_\circ + hF(y_\circ,u))\Big\}$$

Optimal Feedback Design

- **Discrete HJB equation**: for all $y_\circ \in \mathbb{R}^\ell$ and $\beta = e^{-\mu h}$

$$v_h(y_\circ) = \inf_{u\in U} \left\{ \tfrac{h}{2}[L(y_\circ,u) + \beta L(y_\circ + hF(y_\circ,u),u)] + \beta v_h(y_\circ + hF(y_\circ,u)) \right\}$$

- Define

$$S_h(y_\circ) = \operatorname*{argmin}_{u\in U} \left\{ \tfrac{h}{2}[L(y_\circ,u) + \beta L(y_\circ + hF(y_\circ,u),u)] + \beta v(y_\circ + hF(y_\circ,u)) \right\}$$

- **Optimal feedback**: $u_j^* = S_h(y_j^*)$, i.e.,

$$v_h(y_\circ) = J_h(u_h^*;t_\circ,y_\circ), \qquad y_{j+1}^* = y_j^* + hF(y_j^*,S_h(y_j^*)) \text{ for } j \geq 0, \quad y_0^* = y_\circ$$

- **But**: (discrete) HJB is difficult task for $\ell \geq 9$...

Numerical Startegy

- **Optimal control of evolution problems**:
$$\min \mathrm{J}(\mathrm{y},\mathrm{u}) \quad \text{s.t.} \quad \dot{\mathrm{y}}(t) = \mathrm{F}(\mathrm{y}(t),\mathrm{u}(t)) \text{ for } t>0,\ \mathrm{y}(0)=\mathrm{y}_\circ,\ \mathrm{u}\in\mathscr{U}$$
- **Galerkin approximation** with $\psi_1,\ldots,\psi_\ell$:
$$\min J(y,u) \quad \text{s.t.} \quad \dot{y}(t) = F(y(t),u(t)) \text{ for } t>0,\ y(t_\circ)=y_\circ,\ u\in U$$
- **Trapezoidal sum** for J and **Euler's method**
- **Discrete HJB equation**: $\beta = e^{-\mu h}$, for all $y_\circ \in \mathbb{R}^\ell$
$$v_h(y_\circ) = \inf_{u\in U}\left\{\tfrac{h}{2}\big(L(y_\circ,u)+\beta L(y_\circ+hF(y_\circ,u),u)\big)+\beta v_h(y_\circ+hF(y_\circ,u))\right\}$$
- Utilize $\Upsilon_h = [a_1,b_1]\times\ldots\times[a_\ell,b_\ell]\subset\mathbb{R}^\ell$
- Rectlinear partition of Υ_h with vortices $y_1,\ldots,y_M\in\mathbb{R}^\ell$
- Compute piecewise ℓ-linear $v_h^k:\Upsilon_h\to\mathbb{R}$ with
$$v_h^k(y_i) = \inf_{u\in U}\left\{\tfrac{h}{2}\big(L(y_i,u)+\beta L(y_i+hF(y_i,u),u)\big)+\beta v_h^k(y_i+hF(y_i,u))\right\},\quad i=1,\ldots,M$$
- Fixed point method with
    - nested iteration ($h=0.2$, $h/4$, $h/16$)
    - parallelization
- **Current research**: Alla (Hamburg) & Falcone (Rome)

Boundary Control of Burgers' equation

Consider

$$\min J(y,u) = \frac{1}{2}\int_0^\infty \Big(\int_\Omega |y(t,\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x} + \beta\,|u(t)|^2\Big) e^{-\mu t}\,\mathrm{d}t,$$

subject to the Burgers equation

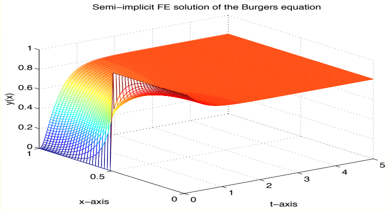
$$\begin{aligned} y_t - \nu y_{xx} + y y_x &= 0 && \text{in } Q\\ \nu y_x(\cdot,0) &= u && \text{in } (0,\infty)\\ \nu y_x(\cdot,1) &= 0 && \text{in } (0,\infty)\\ y(0,\cdot) &= y_\circ && \text{in } \Omega \end{aligned}$$

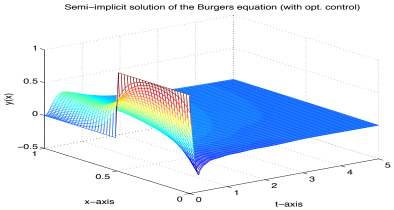
and control constraints

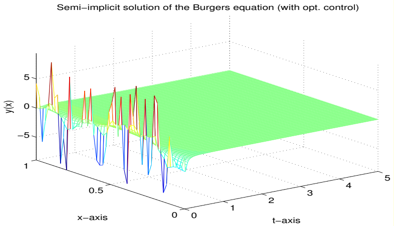
$$\mathscr{U} = \Big\{u \in L^2_{loc}(0,\infty)\ :\ u(t) \in U = [0,1] \text{ for } t \in (0,\infty)\Big\}$$

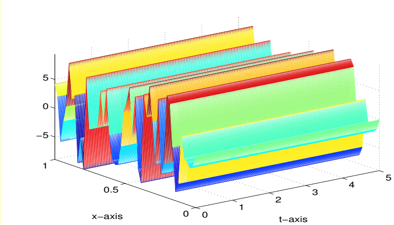
Numerical Results

Discretization: $\ell = 4$ PODs; grid size for $\Upsilon \subset \mathbb{R}^\ell$: $24 \times 16 \times 4 \times 4$









POD Based Model Predictive Control (Alla/V.'14)

Model Predictive Control or Receding horizon Control (Grüne/Pannek'11, Rawlings/Mayne'09)

- **Infinite horizon problem**:

$$\min J(u;t_\circ,y_\circ)=\int_{t_\circ}^{\infty}\ell\big(y_{[u,t_\circ,y_\circ]}(t),u(t)\big)\,\mathrm{d}t \quad \text{s.t.} \quad u\in U_{\mathrm{ad}}(t_\circ)=\{u_a\le v\le u_b \text{ in } [t_\circ,\infty)\}$$

where $\ell(\varphi,v)=(\|\varphi-y_d\|^2+\lambda\,\|v\|^2)/2$ and $y=y_{[u,t_\circ,y_\circ]}$ solves

$$\dot y(t)=\mathscr{F}(y(t),u(t)) \text{ for } t\in(t_\circ,\infty),\quad y(t_\circ)=y_\circ \tag{$*$}$$

- **Goal**: find state feedback $u(t)=\mu(y(t))\in U_{\mathrm{ad}}(t_\circ)$ so that optimal state is given by

$$\dot y(t)=\mathscr{F}(y(t),\mu(y(t))) \text{ for } t\in(t_\circ,\infty),\quad y(t_\circ)=y_\circ$$

- **Model predictive control algorithm**: finite horizon $N\Delta t$ with $N\ge 2$

```
for n = 0,1,2,... do
    Measure the state y(t_n) of the system at t_n = nΔt.
    Set t_∘ = t_n, t_∘^N = t_∘ + NΔt, y_∘ = y(t_n) and compute solution ū^N to
        min Ĵ^N(u; t_∘, y_∘) = ∫_{t_∘}^{t_∘^N} ℓ(y_[u,t_∘,y_∘](t), u(t)) dt  s.t.  u ∈ U_ad^N(t_∘) = {u_a ≤ v ≤ u_b in [t_∘, t_∘^N]}
    where y = y_[u,t_∘,y_∘] solves (*) on [t_∘, t_N]
    Define feedback law μ^N(t; t_∘, y_∘) = ū^N(t), t ∈ (t_∘, t_∘ + Δt].
    Use u = μ^N(·) to compute y by solving (*) on [t_n, t_{n+1}].
end for
```

$$\min \hat J^N(u;t_\circ,y_\circ)=\int_{t_\circ}^{t_\circ^N}\ell\big(y_{[u,t_\circ,y_\circ]}(t),u(t)\big)\,\mathrm{d}t \quad \text{s.t.} \quad u\in U^N_{\mathrm{ad}}(t_\circ)=\{u_a\le v\le u_b \text{ in } [t_\circ,t_\circ^N]\}$$

Model Predictive Control or Receding Horizon Control (Grüne/Pannek'11, Rawlings/Mayne'09)

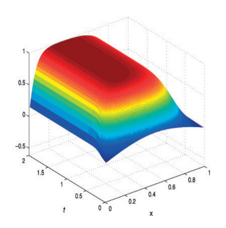
- **Goal**: find state feedback $u(t) = \mu(y(t)) \in U_{\mathrm{ad}}(t_\circ)$ so that optimal state is given by
$$\dot y(t) = \mathscr{F}(y(t), \mu(y(t))) \text{ for } t \in (t_\circ, \infty), \quad y(t_\circ) = y_\circ$$
- **Value functions**:
$$v(t_\circ, y_\circ) = \inf_{u \in U_{\mathrm{ad}}(t_\circ)} \hat J(u; t_\circ, y_\circ) = \inf_{u \in U_{\mathrm{ad}}(t_\circ)} \int_{t_\circ}^{\infty} \ell(y_{[u,t_\circ,y_\circ]}(t), u(t))\,\mathrm{d}t$$
$$v^N(t_\circ, y_\circ) = \inf_{u \in U^N_{\mathrm{ad}}(t_\circ)} \hat J^N(u; t_\circ, y_\circ) = \inf_{u \in U^N_{\mathrm{ad}}(t_\circ)} \int_{t_\circ}^{t_\circ^N} \ell(y_{[u,t_\circ,y_\circ]}(t), u(t))\,\mathrm{d}t$$
- **Asymptotic stability**: y_* equilibrium, i.e., $\mathscr{F}(y_*, \mu(y_*)) = 0$
$$\|y_{[\mu(\cdot),t_\circ,y_\circ]}(t) - y_*\| \to 0 \text{ strictly decreasing } \begin{cases} \text{as } t \to \infty \text{ and} \\ \text{as } \|y_\circ - y_*\| \to 0 \end{cases}$$
- **Relaxed dynamic programming principle**: there is $\alpha^N \in (0,1]$ satisfying
$$v^N(t_\circ, y_\circ) \geq v^N(t_\circ + \Delta t, y_{[\mu^N(\cdot),t_\circ,y_\circ]}(t_\circ + \Delta t)) + \alpha^N \ell(y_\circ, \mu^N(y_\circ)) \tag{1}$$
$\Rightarrow$ asymptotic stability of the feedback law μ^N
- **Computation of N and α^N** (Grüne/Pannek'11): decay of the state, exponential controllability and (simple) feedback law $u = -Ky$ with $K > 0$
- **Reduced-order approach**: control state projection error of reduced-order model

Numerical example (Alla/V.'14)

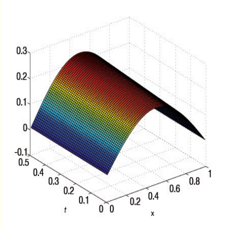
- **Dynamical system**: $(\theta, \rho) = (0.1, 11)$

$$
\begin{aligned}
y_t - \theta y_{xx} + y_x + \rho(y^3 - y) &= u && \text{in } Q\\
y(0,\cdot) = y(1,\cdot) &= 0 && \text{in } (t_\circ, \infty)\\
y(t_\circ) &= y_\circ && \text{in } \Omega = (0,1) \subset \mathbb{R}
\end{aligned}
$$

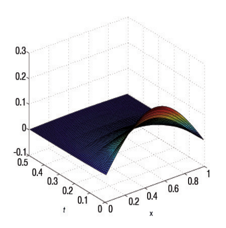
- **Equilibrium and cost**: $y_* \equiv 0$, $\ell(y,u) = \frac{1}{2}\left(\|y\|^2_{L^2(\Omega)} + \sigma\,\|u\|^2_{L^2(\Omega)}\right)$
- **Results**: $N = 10$ and $K = 2.46$, speed-up 13-14 in the control-constrained case



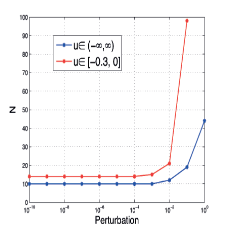
uncontrolled solution



MPC with $N = 3$



MPC with $N = 10$



optimal horizon for given ROM error

Related Literature

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