POD for Nonlinear Systems
Suboptimal Control & Parameter Estimation

Stefan Volkwein

Institute for Mathematics and Scientific Computing
University of Graz, Austria

CEA-EDF-INRIA School

Model reduction: theory and Applications

October 8-10, 2007, Rocquencourt, France

Motivation

- Optimal control of evolution problems:

  $$\min J(y,u) \quad \text{s.t.} \quad \dot{y}(t) = F(y(t),u(t)) \text{ for } t > 0,\ y(0) = y_\circ,\ u \in \mathcal{U}_{ad}$$

- Optimization methods
  - First-order methods: gradient type methods
    $\Rightarrow$ per iteration nonlinear state and linear adjoint equations
  - Second-order methods: SQP or Newton methods
    $\Rightarrow$ per iteration coupled linear state and linear adjoint equations

- Spatial discretization by FE or FD
  $\Rightarrow$ large-scale problems and feedback-strategies not feasible

- Model reduction by POD

Outline

- Suboptimal control:
  - nonlinear heat equation
  - snapshot ensembles in control
- Parameter identification: sampling with respect to the parameters
  - reliable identification by nonlinear optimization
  - bilevel optimization as an application
- A-posteriori error analysis: choose number of POD basis elements
  - error measure via adjoint calculus
  - vary number of POD basis elements
  - no change of POD basis in contrast to OS-POD [Kunisch/V.]
- Static output feedback design (SOF)
- References

Nonlinear heat equation [Diwoky/V.]

- Model problem:

$$\min J(y,u) = \frac{1}{2}\int_\Omega |y(T,x) - z(x)|^2\,\mathrm{d}x + \frac{\beta}{2}\int_0^T\int_\Gamma |u(t,s)|^2\,\mathrm{d}s\mathrm{d}t$$

subject to

$$\begin{aligned} y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x) \in Q = (0,T)\times\Omega \\ \frac{\partial y}{\partial n}(t,s) &= b(y(t,s)) + u(t,s) && \text{for } (t,s) \in \Sigma = (0,T)\times\Gamma \\ y(0,x) &= y_\circ(x) && \text{for } x \in \Omega \subset \mathbb{R}^2 \end{aligned}$$

- Assumptions: $T, \beta, k > 0$, $z, y_\circ \in C(\overline{\Omega})$, $b \in C^{2,1}(\mathbb{R})$ with $b' \leq 0$

Infinite-dimensional problem

- Optimization variables: $z = (y, u) \in Z$, Z function space
- Equality constraints: $e = (e_1, e_2)$

$$
\begin{aligned}
\langle e_1(z), \varphi \rangle = {} & \int_0^T \int_\Omega y_t(t,x)\varphi(t,x) + k\nabla y(t,x) \cdot \nabla \varphi(t,x)\,\mathrm{d}x\mathrm{d}t \\
& - \int_0^T \int_\Gamma \big(b(y(t,s)) + u(t,s)\big)\varphi(t,s)\,\mathrm{d}s\mathrm{d}t \\
e_2(z) = {} & y(0,\cdot) - y_\circ
\end{aligned}
$$

- Infinite-dimensional optimization in function spaces:

$$
\min J(z) \quad \text{subject to} \quad e(z) = 0
$$

- Lagrange function: $L(z, p) = J(z) + \langle e(z), p \rangle$
- Optimality conditions: $\nabla L(z, p) \overset{!}{=} 0$ (Fréchet-derivatives)

First-order optimality conditions

- $\nabla_y L(y,u,p) \stackrel{!}{=} 0$: adjoint equation

$$
\begin{aligned}
-p_t(t,x) &= k\Delta p(t,x) && \text{for } (t,x) \in Q = (0,T)\times\Omega\\
\frac{\partial p}{\partial n}(t,s) &= b'(y(t,s))p(t,s) && \text{for } (t,s) \in \Sigma = (0,T)\times\Gamma\\
p(T,x) &= -(y(T,x) - z(x)) && \text{for } x \in \Omega
\end{aligned}
$$

- $\nabla_u L(z,p) \stackrel{!}{=} 0$: optimality condition $\beta u = kp$ on Σ
- $\nabla_p L(z,p) \stackrel{!}{=} 0$: state equation

$$
\begin{aligned}
y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x) \in Q\\
\frac{\partial y}{\partial n}(t,s) &= b(y(t,s)) + u(t,s) && \text{for } (t,s) \in \Sigma\\
y(0,x) &= y_\circ(x) && \text{for } x \in \Omega
\end{aligned}
$$

SQP methods

- SQP: sequentiel quadratic programming
- Quadratic programming problem: $L(z,p) = J(z) + \langle e(z), p\rangle$

$$\begin{aligned} &\min L(z^n,p^n) + L_z(z^n,p^n)\delta z + \tfrac{1}{2} L_{zz}(z^n,p^n)(\delta z,\delta z) \\ &\text{subject to } e(z^n) + e'(z^n)\delta z = 0 \end{aligned} \qquad (\mathrm{QP}^n)$$

- First-order optimality conditions for (QP^n): KKT system

$$\begin{pmatrix} L_{zz}(z^n,p^n) & e'(z^n)^\star \\ e'(z^n) & 0 \end{pmatrix} \begin{pmatrix} \delta z \\ \delta p \end{pmatrix} = - \begin{pmatrix} L_z(z^n,p^n) \\ e(z^n) \end{pmatrix}$$

- Convergence: locally quadratic rate in (z^n,p^n) (infinite-dimensional)
- Globalization: modification of the Hessian and line-search methods
- Alternative: trust-region methods

POD model reduction

- Goal: POD Galerkin ansatz using ℓ POD basis functions
- Snapshot POD: solve of heat equation for $0 \le t_1 < \ldots < t_n \le T$
- Problems:
  - unknown optimal control $\Rightarrow$ good snapshot set?
  - $u = \frac{k}{\beta} p$ depends on p $\Rightarrow$ POD approximation for p?
- Strategy: iterate basis computation and include adjoint information in the snapshot ensemble

Dynamic POD strategy [Hinze et al./Sachs et al.]

(1) Choose estimate u^0; compute snapshots by solving state equation with $u = u^0$ and adjoint equation with $y = y(u^0)$; $i := 0$

(2) Determine ℓ POD basis functions and associated ROM of infinite-dimensional optimization problem

(3) Compute solution u^{i+1} of optimization problem (e.g., by SQP)

(4) If $\Psi(i) = \dfrac{\|u^{i+1} - u^i\|}{\|u^{i+1}\|} \le TOL$ then stop (stopping criterium)

(5) $i := i + 1$; compute snapshots by solving state equation with control $u = u^i$ and adjoint equation with $y = y(u^i)$; go back to (2)

Alternative: OS-POD, i.e., change of basis within the optimization via optimality conditions

Numerical results

Data: $y_\circ(x_1,x_2) = 10x_1x_2$, $z(x_1,x_2) = 2 + 2|2x_1 - x_2|$, $b(y) = \arctan(y)$, $k = \beta = \frac{1}{10}$, $T = 1$, 185 FEs

Recall: $\Psi(i) = \frac{\|u^{i+1}-u^i\|}{\|u^{i+1}\|}$ stopping criterium for dynamic POD strategy

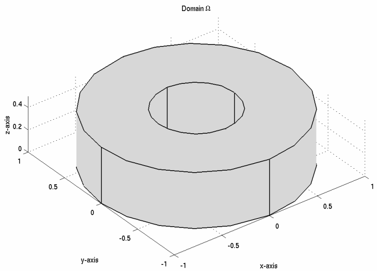
i	relative L^2 error for y	relative L^2 error for u	$J(y,u)$	$\Psi(i)$
0	4.4	12.0	0.358	1.00
1	1.0	8.1	0.360	0.13
2	0.9	6.8	0.361	0.08
POD_{opt}	0.5	5.7	0.358	
FE			0.358	

		POD	FE
Compute snapshots	M-flops	18	
	CPU time in s	3.3	
Compute POD basis	M-flops	0.44	
	CPU time in s	0.01	
Solve with SQP	M-flops	84	
	CPU time in s	22	
total	M-flops	$1.0 \cdot 10^2$	$1.9 \cdot 10^5$
	CPU time in s	$2.5 \cdot 10^1$	$6.6 \cdot 10^3$

Suboptimal control

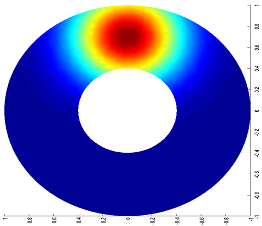
PDE:

$$
\begin{aligned}
y_t &= \Delta y && \text{in } (0,T)\times\Omega \\
\frac{\partial y}{\partial n} &= 0 && \text{on } (0,1)\times\Gamma_1 \\
\frac{\partial y}{\partial n} &= u(t)q && \text{on } (0,1)\times\Gamma_2 \\
y(0) &= 0 && \text{on } \Omega\subset\mathbb{R}^3
\end{aligned}
$$



Boundary condition at $z = 0.5$:

$$
q(t,\mathrm{x}) = e^{-\left(x-0.7\cos(2\pi t)\right)^2} \cdot e^{-\left(y-0.7\sin(2\pi t)\right)^2}
$$



Cost functional:

$$
J(y,u) = \frac{1}{2}\int_\Omega |y(T)-1|^2\,\mathrm{d}x + \frac{\sigma}{2}\int_0^T |u(t)|^2\,\mathrm{d}t
$$

POD computation

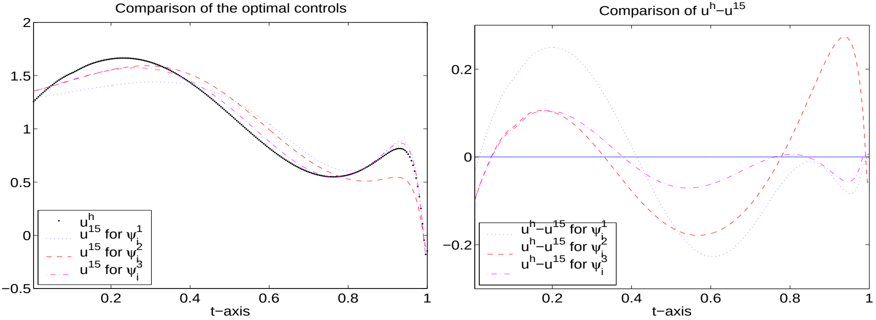
- Snapshot ensembles:

$$\begin{aligned}
\mathcal{V}_1 &= \operatorname{span}\left\{\{\bar{y}^h(t_j)\}_j, \left\{\frac{\bar{y}^h(t_j)-\bar{y}^h(t_{j-1})}{\Delta t}\right\}_j\right\}\\
\mathcal{V}_2 &= \operatorname{span}\left\{\{\bar{p}^h(t_j)\}_j, \left\{\frac{\bar{p}^h(t_j)-\bar{p}^h(t_{j-1})}{\Delta t}\right\}_j\right\}\\
\mathcal{V}_3 &= \mathcal{V}_1 \cup \mathcal{V}_2
\end{aligned}$$

- $E(\ell) = \sum_{i=1}^{\ell} \lambda_i \cdot 100\%$:

ℓ	$E(\ell)$ for $\mathcal{V}^1$	$E(\ell)$ for $\mathcal{V}^2$	$E(\ell)$ for $\mathcal{V}^3$
$\ell = 1$	45.89 %	70.44 %	48.20 %
$\ell = 3$	87.65 %	97.41 %	84.39 %
$\ell = 7$	99.37 %	100.00 %	98.06 %
$\ell = 11$	99.78 %	100.00 %	99.82 %
$\ell = 15$	99.80 %	100.00 %	99.90 %

Approximation of the control variable



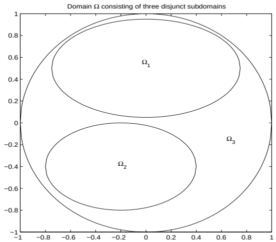
ℓ	$\|u^h - u^\ell\|$ for $\{\psi_i^1\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^2\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^3\}_{i=1}^\ell$
$\ell = 1$	0.5100	0.5437	0.4672
$\ell = 3$	0.3792	0.1200	0.1869
$\ell = 5$	0.3506	0.0588	0.1201
$\ell = 9$	0.3031	0.0585	0.0566
$\ell = 13$	0.2057	0.0596	0.0555

$\|u^h - u^\ell\|$ for different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the ensembles $\mathcal{V}_j$, $j = 1, 2, 3$

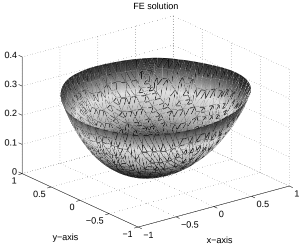
Parameter estimation [Kahlbacher/V., FWF P-19588]

- Model equations: $c = \frac{7}{10}$, $\beta(x) = \begin{pmatrix} 1 \\ x_1 \end{pmatrix}$, $f(x) = x_1$, $\sigma = \frac{3}{2}$, $g \equiv 1$

$$\left.\begin{aligned} -c\,\Delta u + \beta \cdot \nabla u + 10u^3 + au &= f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ c\,\frac{\partial u}{\partial n} + \sigma\, u &= g \quad \text{on } \Gamma \end{aligned}\right. \qquad (*)$$



$a_i = a|_{\Omega_i}$,
$i = 1, 2, 3$



- Data: choose $a_{\mathrm{id}} \geq 0$ and compute (FE) solution $u(a_{\mathrm{id}})$ to $(*)$
- Reconstruction: estimate $a \geq 0$ from $u_d = (1 + \varepsilon\delta)u(a_{\mathrm{id}})\big|_\Gamma$ with random $|\varepsilon| \leq 1$ and factor $\delta = 5\%$

Nonlinear optimization

- Constrained optimization: $\tilde{\kappa}_i = \kappa_i|\Omega_i|$

$$\min J(a,u) = \int_\Gamma \alpha|u-u_d|^2\,\mathrm{d}s + \sum_{i=1}^{3}\tilde{\kappa}_i|a_i|^2 \text{ s.t. } (a,u) \text{ solves PDE \& } a \geq 0$$

- Relaxation of the inequality:

$$\min J^\varrho_\lambda(a,u) = J(a,u) + \tfrac{1}{\varrho}\max\left\{0, \lambda + \varrho(0-a)\right\}^2 \text{ s.t. } (a,u) \text{ solves PDE}$$

- Outer loop: augmented Lagrangian method $\rightarrow$ control of ϱ^k and λ^k
- Inner loop: globalized SQP algorithm with fixed (ϱ^k, λ^k) for

$$\min J^{\varrho^k}_{\lambda^k}(a,u) \quad \text{s.t.} \quad \left\{ \begin{array}{rll} -c\,\Delta u + \beta\cdot\nabla u + 10u^3 + au & = f & \text{in } \Omega \\ c\,\frac{\partial u}{\partial n} + \sigma\, u & = g & \text{on } \Gamma \end{array} \right.$$

Reduced-order modeling

- Model equations: $c = \frac{7}{10}$, $\beta(x) = \begin{pmatrix} 1 \\ x_1 \end{pmatrix}$, $f(x) = x_1$, $\sigma = \frac{3}{2}$, $g \equiv 1$

$$\begin{aligned} -c\,\Delta u + \beta \cdot \nabla u + 10u^3 + au &= f && \text{in } \Omega \subset \mathbb{R}^2 \\ c\,\frac{\partial u}{\partial n} + \sigma\, u &= g && \text{on } \Gamma \end{aligned}$$

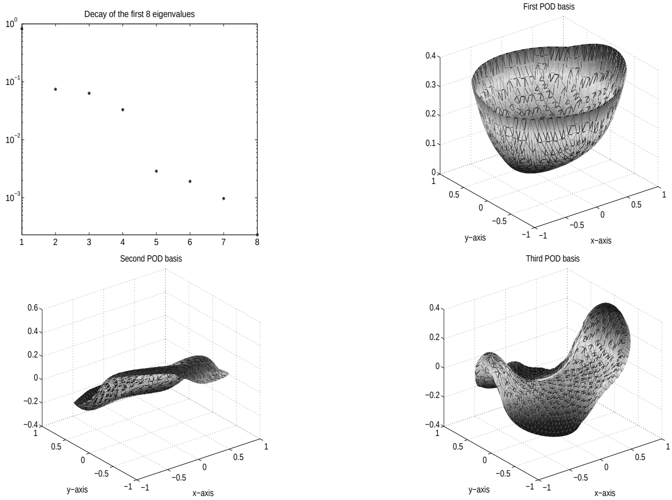
- Snapshots: (FE) solutions $\{u_j\}_{j=1}^n$ for n different a_j
- Parameter grid for a_j: interpolation grid [Maday, Patera, Nguyen,...]
- POD basis of rank $\ell = 8$:

$$\min \sum_{j=1}^{n} \alpha_j \left\| u_j - \sum_{i=1}^{\ell} \langle u_j, \psi_i \rangle\, \psi_i \right\|^2 \quad \text{s.t.} \quad \int_\Omega \psi_i \psi_j \,\mathrm{d}x = \delta_{ij} \qquad (\mathbf{P}^\ell)$$

- Solution to $(\mathbf{P}^\ell)$: correlation matrix $K_{ij} = \int_\Omega u_i u_j \,\mathrm{d}x$

$$K v_i = \lambda_i v_i, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell, \quad \psi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} \alpha_j (v_i)_j u_j$$

POD computation



Choice for the regularization parameter $\kappa = (\kappa_1, \kappa_2, \kappa_3)$

- Initialization: choose POD basis $\psi_1, \ldots, \psi_\ell$
- Bilevel optimization problem:

$$\min_{\kappa_a \le \kappa \le \kappa_b} J(\kappa) = \alpha \int_\Gamma |u_\kappa^\ell - u_d|^2 \,\mathrm{d}s$$

s.t. $(a_\kappa^\ell, u_\kappa^\ell)$ suboptimal POD solution for

$$\begin{cases} \min\limits_{(a,u)} J_\kappa(a,u) = \alpha \int_\Gamma |u - u_d|^2 \,\mathrm{d}s + \sum\limits_{i=1}^{3} \kappa_i \frac{|\Omega_i|}{2} |a_i|^2 \\ \text{s.t. } a \ge 0 \text{ and} \\ \qquad -c\,\Delta u + \beta \cdot \nabla u + 10u^3 + au = f \quad \text{in } \Omega \subset \mathbb{R}^2 \\ \qquad c\,\frac{\partial u}{\partial n} + \sigma\, u = g \quad \text{on } \Gamma \end{cases}$$

- Outer loop: e.g., `fmincon` from MATLAB
- Inner loop: (fast) optimization method based on POD

Numerical results

- Data: $a_{\mathrm{id}} = (6, 7, 8)$, 5% noise
- Starting values: $\kappa^0 = 10^{-4} \cdot (1, 1, 1)$
- Optimal value: $\kappa^* = 10^{-4} \cdot (0.6154, 4.438, 3.221)$
- Results: 1417 FE dof

		rel. error in a
POD, $\ell = 8$	κ^*	1.7%
	$\kappa = \kappa^*/2$	1.9%
	$\kappa = 2\kappa^*$	1.9%
	$\kappa = 10^{-12}$	2.3%
FE	κ^*	1.2%

- CPU time: FE optimization $\sim$ bilevel optimization

A-posteriori error estimates [Tröltzsch/V.]

- Optimal control problem:

$$\min J(y,u) = \frac{1}{2}\,\|y(T)-z\|_H^2 + \frac{\kappa}{2}\int_0^T u(t)^T\mathbf{R}u(t)\,\mathrm{d}t$$

$$\text{s.t.}\quad y_t(t)+\mathcal{A}y(t)=\mathcal{B}u(t)+f(t)\ \text{ in } [0,T] \quad\text{and}\quad y(0)=y_\circ$$

$$u\in U_{ad}=\big\{u\in L^2(0,T)\,|\,u_a\le u\le u_b \text{ in } [0,T]\big\},$$

- H, V Hilbert spaces, $V\hookrightarrow H=H'\hookrightarrow V'$ (e.g., $H=L^2$, $V=H^1$)
- $\mathbf{R}\in\mathbb{R}^{m\times m}$ with $\mathbf{R}\succ 0$, $z\in H$, $\kappa>0$
- $a: V\times V\to\mathbb{R}$ bounded, symmetric, coercive
  $\mathcal{A}: V\to V'$ with $\langle\mathcal{A}\phi,\varphi\rangle_{V',V}=a(\phi,\varphi)$ for all $\phi,\varphi\in V$
- $\mathcal{B}: L^2(0,T)\to L^2(0,T;V')$, $y_\circ\in H$

Optimality conditions

- Optimal solution: (y^*, u^*)
- Adjoint equations:

$$-p^*(t) + \mathcal{A}p^*(t) = 0 \text{ in } [0,T] \quad \text{and} \quad p^*(T) = z - y^*(T)$$

- Variational inequality:

$$\int_0^T \big(\kappa u^* - \mathcal{B}^\star p^*\big)(u - u^*)\,\mathrm{d}t \geq 0 \quad \text{for all } u \in U_{ad}$$

  with adjoint $\mathcal{B}^\star : L^2(0,T;V) \to L^2(0,T)$
- Goal: estimate $\|u^* - u^\ell\|$ for suboptimal $u^\ell \in U_{ad}$
- Idea: there exists $\zeta^\ell \in L^2(0,T)$ satisfying

$$\int_0^T \big(\kappa u^\ell - \mathcal{B}^\star p^\ell + \zeta^\ell\big)(u - u^\ell)\,\mathrm{d}t \geq 0 \quad \text{for all } u \in U_{ad}$$

Error estimate

- Associated state and dual variables: u^ℓ known

$$\begin{aligned} y_t^\ell(t) + \mathcal{A}y^\ell(t) &= \mathcal{B}u^\ell(t) + f(t) && \text{in } [0,T], \quad y^\ell(0) = y_\circ \\ -p_t^\ell(t) + \mathcal{A}p^\ell(t) &= 0 && \text{in } [0,T], \quad p^\ell(T) = z - y^\ell(T) \end{aligned}$$

- Modified variational inequality: $\zeta^\ell \in L^2(0,T)$ satisfies

$$\int_0^T \big(\kappa u^\ell - \mathcal{B}^\star p^\ell + \zeta^\ell\big)(u - u^\ell)\,\mathrm{d}t \geq 0 \quad \text{for all } u \in U_{ad}$$

- Error estimate: $\|u^* - u^\ell\| \leq \frac{1}{\kappa}\|\zeta^\ell\| \to 0$ for $\ell \to \infty$
  $\to$ choose ℓ such that $\|\zeta^\ell\| < \mathrm{TOL}$
- Choice for ζ^ℓ: explicit formula

Numerical example – 1

- Optimal control problem:

$$\min \frac{1}{2}\int_\Omega |y(T,x)-20|^2\,\mathrm{d}x + \frac{5}{2000}\int_0^T |u(t)|^2\,\mathrm{d}t$$

$$\text{s.t.}\quad \begin{cases} y_t - \Delta y = 0 & \text{in } \Omega = (0,1)^2 \subset \mathbb{R}^2 \\ \frac{\partial y}{\partial n} = 0 & \text{on } \Gamma_{N=\{(x_1,x_2)\,|\,x_1\in\{0,1\} \text{ and } x_2\in[0,1]\}} \\ \frac{\partial y}{\partial n} = u(t)b & \text{on } \Gamma\setminus\Gamma_{N=\{(x_1,x_2)\,|\,x_1\in(0,1) \text{ and } x_2\in\{0,1\}\}} \\ -6 \le u \le 1 & \text{in } [0,T] \end{cases}$$

with $b(t,x_1,x_2) = e^{-(x_1-0.7\cos(2\pi t))^2-(x_2-0.7\sin(2\pi t))^2}$

- Optimization method: primal-dual active set strategy
- Discretization: implicit Euler, 1600 FE dofs

Numerical example – 2

ℓ	$\|u^* - u^\ell\|$	$\kappa^{-1}\|\zeta^\ell\|$
2	0.079670	3.361904
3	0.016831	0.065327
4	0.001876	0.002951
5	0.000943	0.002229
6	0.000937	0.002213

FE optimizer	1611 s
Snapshot generation	8 s
POD computation ($\ell = 10$)	5 s
ROM ($\ell = 6$)	$\ll$ 1 s
POD optimizer ($\ell = 6$)	3 s
Computation of ζ^ℓ	18 s

- Example: start with $\ell = 2$ and stop if $\kappa^{-1}\|\zeta^\ell\|_{L^2} < 10^{-2}$ $\longrightarrow$ 50 s

Linear-quadratic-regulator (LQR) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for } t > 0, \quad x(0) = x_\circ$$

with state $x(t) \in \mathbb{R}^\ell$, control $u(t) \in \mathbb{R}^{n_u}$ and $A \in \mathbb{R}^{\ell \times \ell}$, $B \in \mathbb{R}^{\ell \times n_u}$

- Quadratic cost: $J(x, u) = \displaystyle\int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)\,\mathrm{d}t$

with $Q \in \mathbb{R}^{\ell \times \ell}$, $Q \succeq 0$ and $R \in \mathbb{R}^{n_u \times n_u}$, $R \succ 0$

- Goal: (full state) feedback law $u(t) = Fx(t)$ with $F \in \mathbb{R}^{n_u \times \ell}$
- Solution: $F = -R^{-1}B^T P$ with $P = P^T \in \mathbb{R}^{\ell \times \ell}$

$$A^T P + PA + Q - PBR^{-1}B^T P = 0 \qquad \text{(Matrix Riccati)}$$

- Problem: often only partial state measurement available

$\mathcal{H}_2$ static output feedback (SOF) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ$$
$$y(t) = Cx(t)$$

with $A \in \mathrm{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$, $B_1 \in \mathbb{R}^{\ell\times n_w}$, $C \in \mathbb{R}^{n_y\times\ell}$ and

$$x(t) \in \mathbb{R}^\ell, \quad u(t) \in \mathbb{R}^{n_u}, \quad y(t) \in \mathbb{R}^{n_y}, \quad w(t) \in \mathbb{R}^{n_w}$$

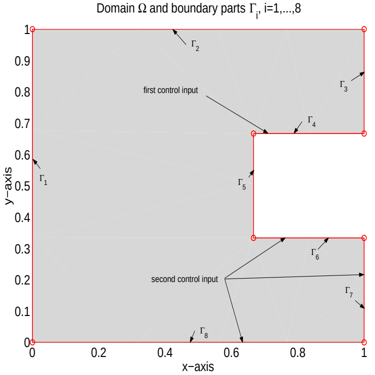
- Feedback law: $u(t) = Fy(t)$ with $F \in \mathbb{R}^{n_u\times n_y}$
- Solution: F given by nonconvex semidefinite programming

$$\min \operatorname{trace}(LB_1B_1^T) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \in \mathbb{R}^{\ell\times\ell} \quad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2\ell\times\ell}$

SOF controller design [Leibfritz/V.]

$$
\begin{aligned}
v_t &= \kappa \Delta v + av && \text{in } \Omega \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (0,T),\ j=1,2,3,5 \\
-\lambda \frac{\partial v}{\partial n} &= \alpha_4 (v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= \hat{\alpha}(v - \hat{c} + \hat{u}(t)) && \text{on } \Gamma_j \times (0,T),\ j=6,7,8 \\
v(0) &= v_\circ && \text{in } \Omega
\end{aligned}
$$



Control: $u(t) \in \mathbb{R}^2,\ n_u = 2$

Measurement: $y(t) \in \mathbb{R}^3,\ n_y = 3$

$$
\begin{aligned}
y_1(t) &= v(0,1;t) \\
y_2(t) &= v(0,0;t) \\
y_3(t) &= v(2/3,1/2;t)
\end{aligned}
$$

Goal: $u(t) = Fy(t),\ F \in \mathbb{R}^{2\times 3}$

Variational form for nonlinear heat equation

- Nonlinear heat equation:

$$
\begin{aligned}
v_t &= \kappa \Delta v + a v && \text{in } \Omega \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (0,T),\ j = 1,2,3,5 \\
-\lambda \frac{\partial v}{\partial n} &= \alpha_4 (v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (0,T) \\
-\lambda \frac{\partial v}{\partial n} &= \hat\alpha (v - \hat c + \hat u(t)) && \text{on } \Gamma_j \times (0,T),\ j = 6,7,8
\end{aligned}
$$

- Variational form: for all $\varphi \in H^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega v_t(t)\varphi + \kappa \nabla v(t) \cdot \nabla \varphi - a v(t) \varphi \,\mathrm{d}x = \kappa \int_\Gamma \frac{\partial v(t)}{\partial n} \varphi \,\mathrm{d}s = \frac{\kappa}{\lambda} \int_\Gamma \lambda \frac{\partial v(t)}{\partial n} \varphi \,\mathrm{d}s \\
&= \frac{\kappa}{\lambda} \int_{\Gamma_4} (\alpha_4 c_4 + \varepsilon_4 \sigma c_4^4)\varphi - (\alpha_4 v(t) + \varepsilon_4 \sigma v^4(t))\varphi - \alpha_4 u_4(t) \varphi \,\mathrm{d}s \\
&\quad + \frac{\kappa}{\lambda} \int_{\Gamma_6 \cup \Gamma_7 \cup \Gamma_8} \hat\alpha \hat c \varphi - \hat\alpha v(t) \varphi - \hat\alpha \hat u(t) \varphi \,\mathrm{d}s
\end{aligned}
$$

$\mathcal{H}_2$ SOF design

- Dynamical system in $\mathbb{R}^N$: spatial discretization (e.g., FE or FD) and linearization

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ$$
$$y(t) = Cx(t)$$

- Goal: feedback law $u(t) = Fy(t)$ with $F \in \mathbb{R}^{2\times 3}$
- Solution: F given by

$$\min \operatorname{trace}\left(LB_1 B_1^T\right) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \qquad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2N\times N}$

- $N = \#$ FE or FD unknowns (!)

Reduced-order model (ROM)

- Compute solution y of nonlinear heat quation with FE or FD at time instances $0 \leq t_1 < \ldots < t_n \leq T$
- Snapshots: $y_j = y(t_j)$ for $i = 1, \ldots, n$
- POD: $\mathcal{R}^n \psi_i = \lambda_i \psi_i$ with $\mathcal{R}^n \psi_i = \sum_{j=1}^{n} \alpha_j \int_\Omega \psi_i y_j \,\mathrm{d}x \; y_j$
- ROM: Galerkin ansatz for nonlinear heat equation with $\psi_1, \ldots, \psi_\ell$

$$\begin{aligned}
\dot{x}(t) &= A^\ell x(t) + G^\ell(x(t)) + B^\ell u(t) + B_1^\ell w(t), \quad x(0) = x_\circ^\ell \\
y(t) &= C^\ell x(t) \\
u(t) &= F^\ell y(t), \quad F^\ell \in \mathbb{R}^{2\times 3}
\end{aligned}$$

Feedback synthesis

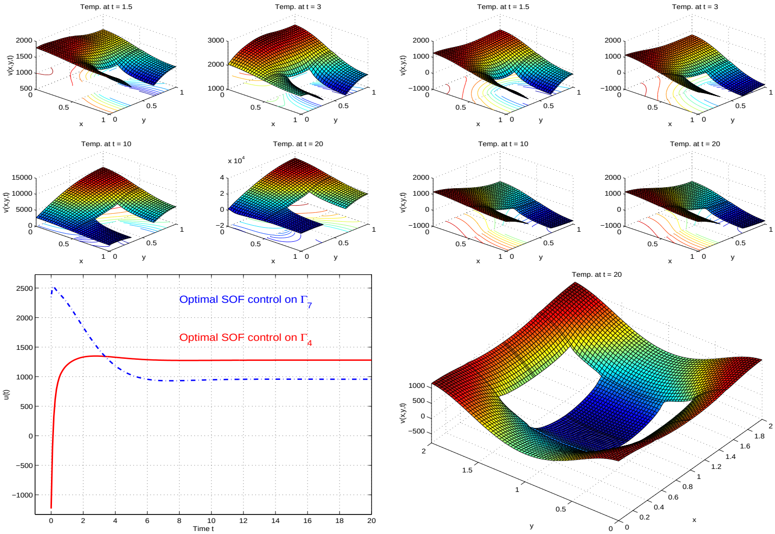
- Reduction in the variable x, not in y and u
- Linearize and set up the SDP problem
  $\Rightarrow$ ℓ is the size of the SDP problem
  $\Rightarrow$ $5 = \ell \ll 3796$ FD unknowns
- Solve SDP by Interior-point trust-region method [Leibfritz/Mostafa]
- Plug in the computed feedback law into the FD modell (closed-loop)

$$\dot{x}(t) = Ax(t) + G(x(t)) + B \underbrace{F^\ell Cx(t)}_{=F^\ell y(t) = u(t)} + B_1 w(t), \; x(0) = x_\circ$$

$$y(t) = Cx(t)$$

$$u(t) = F^\ell y(t) = F^\ell Cx(t)$$

Numerical example (Part 3)



References: suboptimal control & parameter identification

- Diwoky & V.: Nonlinear boundary control for the heat equation utilizing proper orthogonal decomposition, International Series of Numerical Mathematics, 138:73-87, 2001
- Hinze & V.: Error estimates for abstract linear-quadratic optimal control problems using proper orthogonal decomposition, to appear in Computational Optimization and Applications, 2007
- Hinze & V.: Proper Orthogonal Decomposition Surrogate Models for Nonlinear Dynamical Systems: Error Estimates and Suboptimal Control, Lecture Notes in Computational Science and Engineering, 45:261-306, 2005
- Kahlbacher & V.: Model reduction by proper orthogonal decomposition for estimation of scalar parameters in elliptic PDEs, in Proceedings of ECCOMAS CFD, 2006
- Kunisch & V.: Proper orthogonal decomposition for optimality systems, to appear in M^2AN, 2007

References: feedback strategies

- Kunisch & V.: Control of Burgers' equation by a reduced order approach using proper orthogonal decomposition, JOTA, 102:345-371, 1999
- Leibfritz & V.: Numerical feedback controller design for PDE systems using model reduction: techniques and case studies, Proceedings, Santa Fe, USA, 2004
- Leibfritz & V.: Reduced order output feedback control design for PDE systems using proper orthogonal decomposition and nonlinear semidefinite programming, Linear Algebra and Its Applications, 415:542-757, 2006
- Kunisch, V. & Xie: HJB-POD based feedback design for the optimal control of evolution problems, SIADS, 3:701-722, 2004