Suboptimal open-loop control using POD

Stefan Volkwein

Institute for Mathematics and Scientific Computing
University of Graz, Austria

DISC Summer School 2005

Motivation

- Optimal control of evolution problems:

$$\min \mathsf{J}(\mathsf{y},\mathsf{u}) \quad \text{s.t.} \quad \dot{\mathsf{y}}(t) = \mathsf{F}(\mathsf{y}(t),\mathsf{u}(t)) \text{ for } t>0,\; \mathsf{y}(0)=\mathsf{y}_\circ,\; \mathsf{u}\in\mathcal{U}$$

- Optimization methods:
  - First-order methods: gradient type methods
    $\Rightarrow$ per iteration nonlinear state and linear adjoint equations
  - Second-order methods: SQP or Newton methods
    $\Rightarrow$ per iteration coupled linear state and linear adjoint equations
- Spatial discretization by FE or FD
  $\Rightarrow$ large-scale problems and feedback-strategies not feasible
- Model reduction by POD

Outline of the talk

- Suboptimal control of a nonlinear heat equation
- POD error estimate for linear-quadratic optimal control
- Numerical example

Model problem

- Model problem:

$$\min J(y,u) = \frac{1}{2}\int_\Omega |y(T,x)-z(x)|^2\,\mathrm{d}x + \frac{\beta}{2}\int_0^T\int_\Gamma |u(t,s)|^2\,\mathrm{d}s\mathrm{d}t$$

subject to

$$\begin{aligned} y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x)\in Q=(0,T)\times\Omega \\ \frac{\partial y}{\partial n}(t,s) &= b(y(t,s)) + u(t,s) && \text{for } (t,s)\in\Sigma=(0,T)\times\Gamma \\ y(0,x) &= y_\circ(x) && \text{for } x\in\Omega\subsetneq\mathbb{R}^2 \end{aligned}$$

- Assumptions: $T,\beta,k>0$, $z,y_\circ\in C(\overline{\Omega})$, $b\in C^{2,1}(\mathbb{R})$ with $b'\le 0$

Infinite-dimensional problem

- Optimization variables: $z = (y, u) \in Z$, Z function space
- Equality constraints: $e = (e_1, e_2) : Z \to Y$, Y function space

$$\langle e_1(z), \varphi \rangle = \int_0^T \int_\Omega y_t(t,x)\varphi(t,x) + k \nabla y(t,x) \cdot \nabla \varphi(t,x) \, \mathrm{d}x\mathrm{d}t - \int_0^T \int_\Gamma \big(b(y(t,s)) + u(t,s) \big) \varphi(t,s) \, \mathrm{d}s\mathrm{d}t$$

$$e_2(z) = y(0, \cdot) - y_\circ$$

- Infinite-dimensional optimization in function spaces:

$$\min J(z) \quad \text{subject to} \quad e(z) = 0$$

- Lagrange function: $L(z, p) = J(z) + \langle e(z), p \rangle_Y$
- Optimality conditions: $\nabla L(z, p) = 0 \in Z \times Y$ (Fréchet-derivatives)

First-order optimality conditions

- $\nabla_y L(y,u,p) \stackrel{!}{=} 0$: adjoint equation

$$\begin{aligned} -p_t(t,x) &= k\Delta p(t,x) && \text{for } (t,x)\in Q=(0,T)\times\Omega \\ \frac{\partial p}{\partial n}(t,s) &= b'(y(t,s))p(t,s) && \text{for } (t,s)\in \Sigma=(0,T)\times\Gamma \\ p(T,x) &= -(y(T,x)-z(x)) && \text{for } x\in\Omega \end{aligned}$$

- $\nabla_u L(z,p) \stackrel{!}{=} 0$: optimality condition $\beta u = kp$ on Σ
- $\nabla_p L(z,p) \stackrel{!}{=} 0$: state equation

$$\begin{aligned} y_t(t,x) &= k\Delta y(t,x) && \text{for } (t,x)\in Q \\ \frac{\partial y}{\partial n}(t,s) &= b(y(t,s))+u(t,s) && \text{for } (t,s)\in\Sigma \\ y(0,x) &= y_\circ(x) && \text{for } x\in\Omega \end{aligned}$$

SQP methods

- SQP: sequentiel quadratic programming
- Quadratic programming problem: $L(z,p) = J(z) + \langle e(z), p \rangle$

$$\begin{aligned} &\min L(z^n,p^n) + L_z(z^n,p^n)\delta z + \tfrac{1}{2} L_{zz}(z^n,p^n)(\delta z, \delta z) \\ &\text{subject to } e(z^n) + e'(z^n)\delta z = 0 \end{aligned} \qquad (\mathrm{QP}^n)$$

- Optimality conditions for (QP^n): KKT system

$$\begin{pmatrix} L_{zz}(z^n,p^n) & e'(z^n)^\star \\ e'(z^n) & 0 \end{pmatrix} \begin{pmatrix} \delta z \\ \delta p \end{pmatrix} = - \begin{pmatrix} L_z(z^n,p^n) \\ e(z^n) \end{pmatrix}$$

- Convergence: locally quadratic rate in (z^n, p^n) (infinite-dimensional)
- Globalization: modification of the Hessian and line-search methods
- Alternative: trust-region methods

POD model reduction

- Goal: POD Galerkin ansatz using ℓ POD basis functions
- Snapshot POD: solve of heat equation for $0 \leq t_1 < \ldots < t_n \leq T$
- Problems:
  - unknown optimal control $\Rightarrow$ good snapshot set?
  - $u = \frac{k}{\beta} p$ depends on $p \Rightarrow$ POD approximation for p?
- Strategy: iterate basis computation and include adjoint information in the snapshot ensemble

Dynamic POD strategy [Hinze et al./Sachs et al.]

- (1) Choose estimate u^0; compute snapshots by solving state equation with $u = u^0$ and adjoint equation with $y = y(u^0)$; $i := 0$
- (2) Determine ℓ POD basis functions and associated ROM of infinite-dimensional optimization problem
- (3) Compute solution u^{i+1} of optimization problem (e.g., by SQP)
- (4) If $\Psi(i) = \dfrac{\|u^{i+1} - u^i\|}{\|u^{i+1}\|} \leq TOL$ then stop (stopping criterium)
- (5) $i := i + 1$; compute snapshots by solving state equation with control $u = u^i$ and adjoint equation with $y = y(u^i)$; go back to (2)

Numerical results [Diwoky/V.]

Data: $y_\circ(x_1, x_2) = 10x_1 x_2$, $z(x_1, x_2) = 2 + 2|2x_1 - x_2|$, $b(y) = \arctan(y)$, $k = \beta = \frac{1}{10}$, $T = 1$, 185 FEs

Recall: $\Psi(i) = \frac{\|u^{i+1} - u^i\|}{\|u^{i+1}\|}$ stopping criterium for dynamic POD strategy

i	relative L^2 error for y	relative L^2 error for u	$J(y, u)$	$\Psi(i)$
0	4.4	12.0	0.358	1.00
1	1.0	8.1	0.360	0.13
2	0.9	6.8	0.361	0.08
POD_{opt}	0.5	5.7	0.358	
FE			0.358	

		POD	FE
Compute snapshots	M-flops	18	
	CPU time in s	3.3	
Compute POD basis	M-flops	0.44	
	CPU time in s	0.01	
Solve with SQP	M-flops	84	
	CPU time in s	22	
total	M-flops	$1.0 \cdot 10^2$	$1.9 \cdot 10^5$
	CPU time in s	$2.5 \cdot 10^1$	$6.6 \cdot 10^3$

State equation

- V, H real separable Hilbert spaces with $V \hookrightarrow H = H' \hookrightarrow V'$ with $V' =$ set all bounded and linear functions $\chi : V \to \mathbb{R}$
- $a : V \times V \to \mathbb{R}$ bounded, symmetric, coercive
- State equation:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle y(t), \varphi \rangle_H + a(y(t), \varphi) &= \langle (\mathcal{B}u)(t), \varphi \rangle_{V',V}, & t \in [0,T],\ \varphi \in V \\ \langle y(0), \varphi \rangle_H &= \langle y_0, \varphi \rangle_H, & \varphi \in V \end{aligned}$$

$u \in \mathcal{U}$, $\mathcal{U} = \mathcal{U}'$ Hilbert space, $\mathcal{B} \in \mathcal{L}(\mathcal{U}, L^2(0,T;V'))$, $y_0 \in H$

- $\exists! y \in W(0,T) = \{\varphi \in L^2(0,T;V) \,|\, \varphi_t \in L^2(0,T;V')\}$

Optimal control problem

- Quadratic cost functional: $z \in H$, $\sigma > 0$

$$J(y,u) = \frac{1}{2}\left\|y(T) - z\right\|_H^2 + \frac{\sigma}{2}\left\|u\right\|_{\mathcal{U}}^2$$

- Control constraints: $\mathcal{U}_{\mathrm{ad}} \subset \mathcal{U}$ closed, convex, nonempty
- State system:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle y(t),\varphi\rangle_H + a(y(t),\varphi) &= \langle (\mathcal{B}u)(t),\varphi\rangle_{V',V}, & t \in [0,T],\ \varphi \in V \\ \langle y(0),\varphi\rangle_H &= \langle y_0,\varphi\rangle_H, & \varphi \in V \end{aligned}$$

- Linear-quadratic control problem: $z = (y,u)$

$$\min J(z) \quad \text{s.t.} \quad z \in W(0,T) \times \mathcal{U}_{\mathrm{ad}} \text{ solves state system} \qquad \text{(P)}$$

- Unique optimal solution $\bar{z} = (\bar{y},\bar{u})$ to (P)

Optimality conditions

- Dual equations:

$$
\begin{aligned}
-\tfrac{\mathrm{d}}{\mathrm{d}t}\langle \bar p(t), \varphi\rangle_H + a(\bar p(t), \varphi) &= 0, \qquad t \in [0,T],\ \varphi \in V\\
\langle \bar p(T), \varphi\rangle_H + \langle \bar y(T) - z, \varphi\rangle_H &= 0, \qquad \varphi \in V
\end{aligned}
$$

- Reduced problem:

$$
\min \hat J(u) = J(y(u), u) \quad \text{s.t.} \quad u \in \mathcal{U}_{\text{ad}} \qquad (\hat{\mathrm{P}})
$$

- Reduced gradient: $G(u) := \hat J'(u) = \sigma u - \mathcal{B}^\star p(y(u))$

  $\mathcal{B}^\star$ dual operator of $\mathcal{B}$

- Optimality condition: $\langle G(\bar u), u - \bar u\rangle_{\mathcal{U}} \ge 0$ for all $u \in \mathcal{U}_{\text{ad}}$

Continuous POD

- $y(u) \in C^1([0,T];V)$ solution of the state system for $u \in \mathcal{U}$
- Goal: orthogonal basis $\psi_1, \ldots, \psi_\ell$ in V for the space

$$\mathcal{V} = \big\{ y(t) \,|\, t \in [0,T] \big\} \cup \big\{ \dot{y}(t) \,|\, t \in [0,T] \big\}$$

  spanned by $t \mapsto y(t) \in V$ and $t \mapsto \dot{y}(t) = (-Ay + \mathcal{B}u)(t) \in V$

- POD criterium:

$$\min \int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i \rangle_V \, \psi_i \Big\|_V^2 + \Big\| \dot{y}(t) - \sum_{i=1}^{\ell} \langle \dot{y}(t), \psi_i \rangle_V \, \psi_i \Big\|_V^2 \, \mathrm{d}t$$

$$= \sum_{i=\ell+1}^{\infty} \int_0^T \big| \langle y(t), \psi_i \rangle_V \big|^2 + \big| \langle \dot{y}(t), \psi_i \rangle_V \big|^2 \, \mathrm{d}t$$

$$\text{s.t. } \langle \psi_j, \psi_i \rangle_V = \delta_{ij} \quad \text{for } 1 \le i,j \le \ell$$

Computation of the POD basis

- Linear and compact integral operator:

$$\mathcal{R}\psi = \int_0^T \langle \psi, y(t) \rangle_V \; y(t) + \langle \psi, \dot{y}(t) \rangle_V \; \dot{y}(t) \, \mathrm{d}t \quad \text{for } \psi \in V$$

- Eigenvalue problem for the POD basis functions:

$$\mathcal{R}\psi_i^\infty = \lambda_i^\infty \psi_i^\infty, \quad \lambda_1^\infty \geq \lambda_2^\infty \geq \ldots, \quad \text{and } \lambda_i^\infty \to 0 \text{ as } i \to \infty$$

- Error formula: $\mathcal{P}^\ell \varphi = \sum_{i=1}^{\ell} \langle \varphi, \psi_i^\infty \rangle_V \, \psi_i^\infty$ for $\varphi \in V$

$$\int_0^T \Big\| y(t) - \sum_{i=1}^{\ell} \langle y(t), \psi_i^\infty \rangle_V \; \psi_i^\infty \Big\|_V^2 + \Big\| \dot{y}(t) - \sum_{i=1}^{\ell} \langle \dot{y}(t), \psi_i^\infty \rangle_V \; \psi_i^\infty \Big\|_V^2 \mathrm{d}t = \sum_{i=\ell+1}^{\infty} \lambda_i^\infty$$

POD Galerkin schemes

- Discrete state equations: $V^\ell = \operatorname{span}\{\psi_1^\infty, \ldots, \psi_\ell^\infty\}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\langle y^\ell(t),\psi\rangle_H + a(y^\ell(t),\psi) &= \langle(\mathcal{B}u)(t),\psi\rangle_{V',V}, && t\in[0,T],\ \psi\in V^\ell \\ \langle y^\ell(0),\psi\rangle_H &= \langle y_0,\psi\rangle_H, && \psi\in V^\ell \end{aligned}$$

- Discrete dual equations:

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\langle p^\ell(t),\psi\rangle_H + a(p^\ell(t),\psi) &= 0, && t\in[0,T],\ \psi\in V^\ell \\ \langle p^\ell(T),\psi\rangle_H + \langle y^\ell(T)-z,\psi\rangle_H &= 0, && \psi\in V^\ell \end{aligned}$$

POD approximation of $(\hat{\mathrm{P}})$

- Recall $G(u) = \sigma u - \mathcal{B}^\star p$ for $u \in \mathcal{U}$ with $p = p(y(u))$
- Define $G^\ell(u) = \sigma u - \mathcal{B}^\star p^\ell$ for $u \in \mathcal{U}$ with $p^\ell = p^\ell(y^\ell(u))$
- Discrete optimality condition:

$$\langle G^\ell(\bar{u}^\ell), u - \bar{u}^\ell \rangle_{\mathcal{U}} \geq 0 \quad \text{for all } u \in \mathcal{U}_{\mathsf{ad}}$$

- Discrete linear-quadratic control problem:

$$\min \hat{J}^\ell(u) = J(y^\ell(u), u) \quad \text{s.t.} \quad u \in \mathcal{U}_{\mathsf{ad}} \qquad (\hat{\mathrm{P}}^\ell)$$

POD error estimate [Hinze/V.]

- $\bar{u} \in \mathcal{U}_{\text{ad}}$ (unique) linear-quadratic optimal control
- $u^\ell \in \mathcal{U}_{\text{ad}}$ (unique) suboptimal linear-quadratic POD control
- $\bar{y} = \bar{y}(\bar{u}) \in W(0,T)$ associated optimal state
- $\bar{p} = \bar{p}(\bar{y}(\bar{u})) \in W(0,T)$ associated dual
- $\{\psi_i^\infty\}_{i=1}^\ell$ POD basis for $\{\bar{y}(t) \,|\, t \in [0,T]\} \cup \{\bar{y}_t(t) \,|\, t \in [0,T]\}$
- $\bar{y}^\ell = \bar{y}^\ell(\bar{u}) \in W(0,T)$ associated optimal state
- $\bar{p}^\ell = \bar{p}^\ell(\bar{y}^\ell(\bar{u})) \in W(0,T)$ associated dual
- POD estimate for the controls:

$$\|\bar{u} - u^\ell\|_{\mathcal{U}} \le \frac{1}{\sigma} \|\mathcal{B}^\star(\bar{p}^\ell - \bar{p})\|_{\mathcal{U}} \le C\,\|\bar{p}^\ell - \bar{p}\|_{L^2(0,T;V)}$$

POD estimates for state & dual

- Orthogonal projection $\mathcal{P}^\ell : V \to V^\ell = \mathrm{span}\,\{\psi_1^\infty, \ldots, \psi_\ell^\infty\}$:

$$\mathcal{P}^\ell \varphi = \sum_{i=1}^{\ell} \langle \varphi, \psi_i^\infty \rangle_V \, \psi_i^\infty \quad \text{for } \varphi \in V$$

- $y = y(u)$, $y^\ell = y^\ell(u)$ and $p = p(y(u))$, $p^\ell = p^\ell(y^\ell(u))$
- POD state estimate: $X = L^\infty(0,T;H) \cap L^2(0,T;V)$

$$\|y^\ell - y\|_X^2 \le C\Big(\|y_0 - \mathcal{P}^\ell y_0\|_H^2 + \sum_{i=\ell+1}^{\infty} \lambda_i^\infty\Big)$$

- POD dual estimate: $Y = H^1(0,T;V)$

$$\|p^\ell - p\|_{L^2(0,T;V)} \le C\Big(\|y^\ell - y\|_X + \|\mathcal{P}^\ell p - p\|_Y\Big)$$

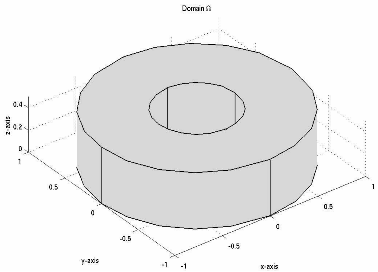
Numerical example (Part 1)

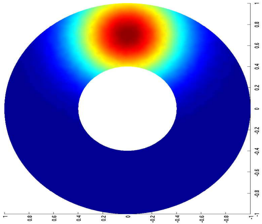
PDE:

$$
\begin{aligned}
y_t &= \Delta y && \text{in } (0,T)\times\Omega \\
\frac{\partial y}{\partial n} &= 0 && \text{on } (0,1)\times\Gamma_1 \\
\frac{\partial y}{\partial n} &= u(t)q && \text{on } (0,1)\times\Gamma_2 \\
y(0) &= 0 && \text{on } \Omega\subset\mathbb{R}^3
\end{aligned}
$$

Boundary condition at $z = 0.5$:

$$
\begin{aligned}
&q(t,\mathrm{x}) \\
&= \exp\big(-[x-0.7\cos(2\pi t)]^2\big) \\
&\quad\cdot\exp\big(-[y-0.7\sin(2\pi t)]^2\big)
\end{aligned}
$$





Numerical example (Part 2)

- Snapshot ensembles:

$$\mathcal{V}_1 = \text{span}\left\{\{\bar{y}^h(t_j)\}_j, \left\{\frac{\bar{y}^h(t_j)-\bar{y}^h(t_{j-1})}{\Delta t}\right\}_j\right\}$$
$$\mathcal{V}_2 = \text{span}\left\{\{\bar{p}^h(t_j)\}_j, \left\{\frac{\bar{p}^h(t_j)-\bar{p}^h(t_{j-1})}{\Delta t}\right\}_j\right\}$$
$$\mathcal{V}_3 = \mathcal{V}_1 \cup \mathcal{V}_2$$

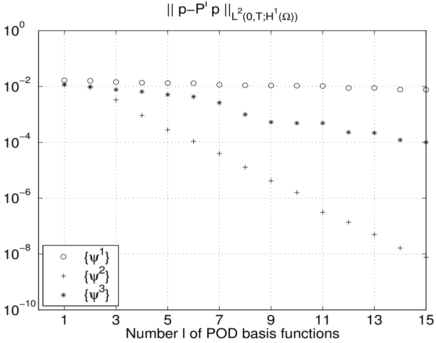
- $E(\ell) = \sum_{i=1}^{\ell} \lambda_i \cdot 100\%$:

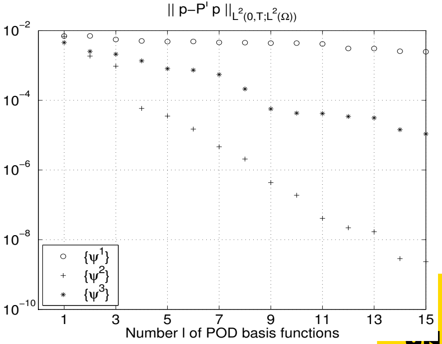
ℓ	$E(\ell)$ for $\mathcal{V}^1$	$E(\ell)$ for $\mathcal{V}^2$	$E(\ell)$ for $\mathcal{V}^3$
$\ell = 1$	45.89 %	70.44 %	48.20 %
$\ell = 3$	87.65 %	97.41 %	84.39 %
$\ell = 7$	99.37 %	100.00 %	98.06 %
$\ell = 11$	99.78 %	100.00 %	99.82 %
$\ell = 15$	99.80 %	100.00 %	99.90 %

Numerical example (Part 3)

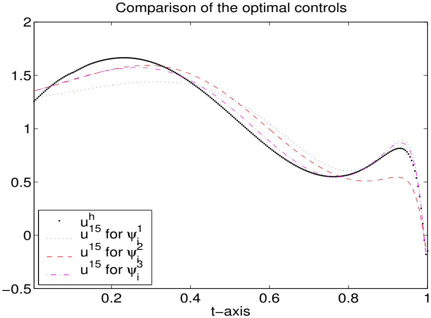
POD dual estimate:

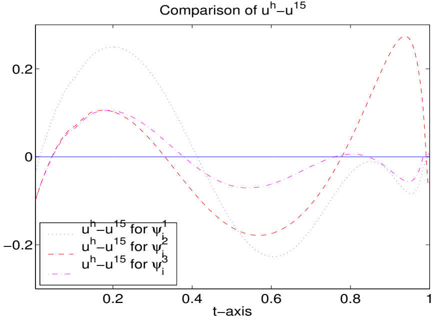
$$\|p^\ell - p\|_{L^2(0,T;V)} \le C\Big(\|y^\ell - y\|_X + \|p - \mathcal{P}^\ell p\|\Big)$$





Numerical example (Part 4)





ℓ	$\|u^h - u^\ell\|$ for $\{\psi_i^1\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^2\}_{i=1}^\ell$	$\|u^h - u^\ell\|$ for $\{\psi_i^3\}_{i=1}^\ell$
$\ell = 1$	0.5100	0.5437	0.4672
$\ell = 3$	0.3792	0.1200	0.1869
$\ell = 5$	0.3506	0.0588	0.1201
$\ell = 9$	0.3031	0.0585	0.0566
$\ell = 13$	0.2057	0.0596	0.0555

$\|u^h - u^\ell\|$ for different POD basis $\{\psi_i^j\}_{i=1}^\ell$ corresponding to the ensembles $\mathcal{V}_j$, $j = 1, 2, 3$

References

- ...
- Diwoky & V.: Nonlinear boundary control for the heat equation utilizing proper orthogonal decomposition, International Series of Numerical Mathematics, 138:73-87, 2001
- V.: Optimal control of a phase-field model using proper orthogonal decomposition, ZAMM, 81:83-97, 2001
- V.: Boundary control of the Burgers equation: optimality conditions and reduced-order approach, International Series of Numerical Mathematics, 139:267-278, 2001
- Hinze & V.: Error estimates for abstract linear-quadratic optimal control problems using proper orthogonal decomposition, (will be) submitted