Feedback strategies

Stefan Volkwein

Institute for Mathematics and Scientific Computing
University of Graz, Austria

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Open and closed loop control

- Open loop control:

$$\text{input } u(t) \rightarrow \boxed{\text{plant}} \rightarrow \text{state } x(t) \in \mathbb{R}^\ell$$

- Closed loop control: determine mapping $\mathcal{F}$ such that

$$u(t) = \mathcal{F}(t, x(t)) \qquad \text{(feedback law)}$$

- Linear-quadratic case: LQR and LQG design
- Nonlinear case: Hamilton-Jacobi-Bellman (HJB) equations

$$v_t(t,x) + H(\nabla_x v(t,x), x) = 0 \quad \text{for } (t,x) \in (0,T) \times \mathbb{R}^\ell$$

Outline of the talk

- Static output feedback (SOF) design
- Numerical example: nonlinear heat equation
- Nonlinear feedback using the HJB equations

Linear-quadratic-regulator (LQR) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) \text{ for } t > 0, \quad x(0) = x_\circ$$

with state $x(t) \in \mathbb{R}^\ell$, control $u(t) \in \mathbb{R}^{n_u}$ and $A \in \mathbb{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$

- Cost: $J(x,u) = \int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)\,\mathrm{d}t$

with $Q \in \mathbb{R}^{\ell\times\ell}$, $Q \succeq 0$ and $R \in \mathbb{R}^{n_u\times n_u}$, $R \succ 0$

- Goal: (full state) feedback law $u(t) = Fx(t)$ with $F \in \mathbb{R}^{n_u\times\ell}$
- Solution: $F = -R^{-1}B^TP$ with $P = P^T \in \mathrm{R}^{\ell\times\ell}$

$$A^TP + PA + Q - PBR^{-1}B^TP = 0 \qquad \text{(Matrix Riccati)}$$

- Problem: often only partial state measurement available

$\mathcal{H}_2$ static output feedback (SOF) design

- Linear dynamical system in $\mathbb{R}^\ell$:

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ$$
$$y(t) = Cx(t)$$

with $A \in \mathrm{R}^{\ell\times\ell}$, $B \in \mathbb{R}^{\ell\times n_u}$, $B_1 \in \mathbb{R}^{\ell\times n_w}$, $C \in \mathbb{R}^{n_y\times\ell}$ and

$$x(t) \in \mathbb{R}^\ell, \quad u(t) \in \mathbb{R}^{n_u}, \quad y(t) \in \mathbb{R}^{n_y}, \quad w(t) \in \mathbb{R}^{n_w}$$

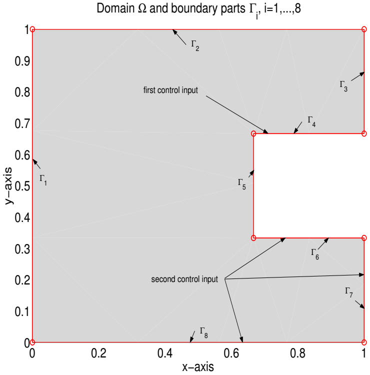
- Feedback law: $u(t) = Fy(t)$ with $F \in \mathbb{R}^{n_u\times n_y}$
- Solution: F given by nonconvex semidefinite programming

$$\min \operatorname{trace}\left(LB_1B_1^T\right) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \in \mathbb{R}^{\ell\times\ell} \quad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2\ell\times\ell}$

SOF controller design [Leibfritz/V.]

$$
\begin{aligned}
v_t &= \kappa\Delta v + av && \text{in } \Omega\times(0,T)\\
-\lambda\frac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j\times(0,T),\ j=1,2,3,5\\
-\lambda\frac{\partial v}{\partial n} &= \alpha_4(v-c_4+u_4(t))+\varepsilon_4\sigma(v^4-c_4^4) && \text{on } \Gamma_4\times(0,T)\\
-\lambda\frac{\partial v}{\partial n} &= \hat\alpha(v-\hat c+\hat u(t)) && \text{on } \Gamma_j\times(0,T),\ j=6,7,8\\
v(0) &= v_\circ && \text{in } \Omega
\end{aligned}
$$



Control: $u(t)\in\mathbb{R}^2$, $n_u=2$

Measurement: $y(t)\in\mathbb{R}^3$, $n_y=3$

$$
\begin{aligned}
y_1(t) &= v(0,1;t)\\
y_2(t) &= v(0,0;t)\\
y_3(t) &= v(2/3,1/2;t)
\end{aligned}
$$

Goal: $u(t)=Fy(t)$, $F\in\mathbb{R}^{2\times 3}$

Variational form for nonlinear heat equation

- Nonlinear heat equation:

$$
\begin{aligned}
v_t &= \kappa \Delta v + av && \text{in } \Omega \times (0,T)\\
-\lambda \tfrac{\partial v}{\partial n} &= 0 && \text{on } \Gamma_j \times (0,T),\ j = 1,2,3,5\\
-\lambda \tfrac{\partial v}{\partial n} &= \alpha_4(v - c_4 + u_4(t)) + \varepsilon_4 \sigma (v^4 - c_4^4) && \text{on } \Gamma_4 \times (0,T)\\
-\lambda \tfrac{\partial v}{\partial n} &= \hat\alpha(v - \hat c + \hat u(t)) && \text{on } \Gamma_j \times (0,T),\ j = 6,7,8
\end{aligned}
$$

- Variational form: for all $\varphi \in H^1(\Omega)$

$$
\begin{aligned}
&\int_\Omega v_t(t)\varphi + \kappa \nabla v(t)\cdot\nabla\varphi - av(t)\varphi\,\mathrm{d}x = \kappa \int_\Gamma \frac{\partial v(t)}{\partial n}\varphi\,\mathrm{d}s = \frac{\kappa}{\lambda}\int_\Gamma \lambda \frac{\partial v(t)}{\partial n}\varphi\,\mathrm{d}s\\
&= \frac{\kappa}{\lambda}\int_{\Gamma_4} \big(\alpha_4 c_4 + \varepsilon_4\sigma c_4^4\big)\varphi - \big(\alpha_4 v(t) + \varepsilon_4 \sigma v^4(t)\big)\varphi - \alpha_4 u_4(t)\varphi\,\mathrm{d}s\\
&\quad + \frac{\kappa}{\lambda}\int_{\Gamma_6\cup\Gamma_7\cup\Gamma_8} \hat\alpha\hat c\varphi - \hat\alpha v(t)\varphi - \hat\alpha\hat u(t)\varphi\,\mathrm{d}s
\end{aligned}
$$

$\mathcal{H}_2$ SOF design

- Dynamical system in $\mathbb{R}^N$: spatial discretization (e.g., FE or FD) and linearization

$$\dot{x}(t) = Ax(t) + Bu(t) + B_1 w(t) \text{ for } t > 0, \qquad x(0) = x_\circ$$
$$y(t) = Cx(t)$$

- Goal: feedback law $u(t) = Fy(t)$ with $F \in \mathbb{R}^{2\times 3}$
- Solution: F given by

$$\min \operatorname{trace}\left(LB_1B_1^T\right) \quad \text{s.t.} \quad H(F,L,V) = 0 \;\&\; V \succ 0 \qquad \text{(SDP)}$$

with $H(F,L,V) = \begin{pmatrix} A(F)^T L + LA(F) + C(F)^T C(F) \\ A(F)^T V + VA(F) + I \end{pmatrix} \in \mathbb{R}^{2N\times N}$

- $N = \#$ FE or FD unknowns (!)

Reduced-order model (ROM)

- Compute solution y of nonlinear heat quation with FE or FD at time instances $0 \leq t_1 < \ldots < t_n \leq T$
- Snapshots: $y_j = y(t_j) \in H = L^2(\Omega)$ for $i = 1, \ldots, n$
- POD: $\mathcal{R}^n \psi_i = \lambda_i \psi_i$ with $\mathcal{R} = \sum_{j=1}^{n} \alpha_j \langle \cdot, y_j \rangle_H y_j$
- ROM: Galerkin ansatz for nonlinear heat equation with $\psi_1, \ldots, \psi_\ell$

$$
\begin{aligned}
\dot{x}(t) &= A^\ell x(t) + G^\ell(x(t)) + B^\ell u(t) + B_1^\ell w(t), \quad x(0) = x_\circ^\ell \\
y(t) &= C^\ell x(t) \\
u(t) &= F^\ell y(t), \quad F^\ell \in \mathbb{R}^{2\times 3}
\end{aligned}
$$

Feedback synthesis

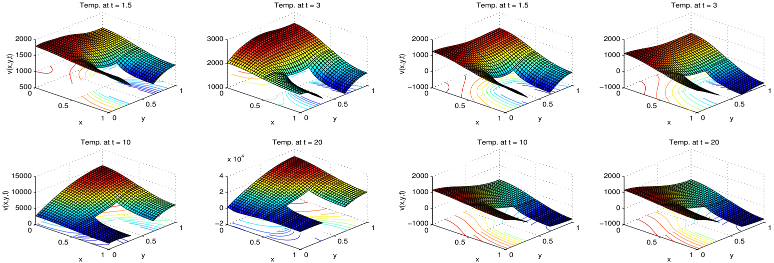
- Reduction in the variable x, not in y and u
- Linearize and set up the SDP problem
  $\Rightarrow \ell$ is the size of the SDP problem
  $\Rightarrow 5 = \ell \ll 3796$ FD unknowns
- Solve SDP by Interior-point trust-region method [Leibfritz/Mostafa]
- Plug in the computed feedback law into the FD modell (closed-loop)

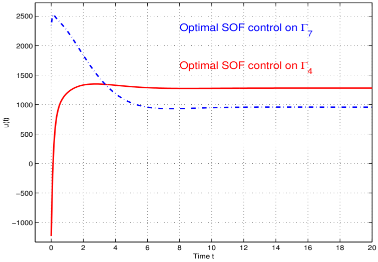
$$\dot{x}(t) = Ax(t) + G(x(t)) + B \underbrace{F^\ell Cx(t)}_{=F^\ell y(t) = u(t)} + B_1 w(t), \; x(0) = x_\circ$$

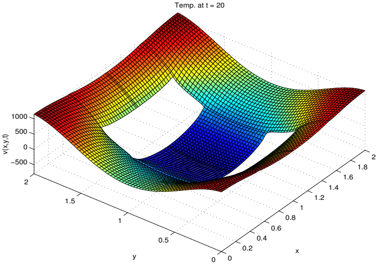
$$y(t) = Cx(t)$$

$$u(t) = F^\ell y(t) = F^\ell Cx(t)$$

Numerical example (Part 3)







Hamilton-Jacobi-Bellman (Part 1)

- Dynamical system in $\mathbb{R}^\ell$:

$$\begin{cases} \dot{y}(t) = F(y(t), u(t)) & \text{for } t > 0 \\ y(0) = y_\circ \end{cases} \tag{1}$$

- Admissible controls: $u \in L^2(0, T; \mathbb{R}^m)$, $u(t) \in U \subset \mathbb{R}^m$
- Cost:

$$\min J(u; y_\circ) = \int_0^\infty L(y(t), u(t)) e^{-\mu t} \, \mathrm{d}t$$

with $y = y(u)$ solution to (1) and $\mu > 0$

Hamilton-Jacobi-Bellman (Part 2)

- Euler's method: $y_{j+1} = y_j + hF(y_j, u_j)$ for $j \geq 0$
- Discrete cost:

$$J_h(u; y_\circ) = \frac{h}{2}\Big(L(y_\circ, u_0) + \sum_{j=1}^{\infty} e^{-\mu jh}\left[L(y_j, u_{j-1}) + L(y_j, u_j)\right]\Big)$$

- Discrete minimal value function:

$$v_h(y_\circ) = \inf\{J_h(u_h; y_\circ) : u_h \in \mathcal{U}^h\}$$

with $\mathcal{U}^h = \{u_h = \{u_0, u_1, \ldots\} \mid u_j \in U\}$

- Discrete HJB equation: for all $y_\circ \in \mathbb{R}^\ell$ and $\beta = e^{-\mu h}$

$$v_h(y_\circ) = \inf_{u \in U} \left\{\frac{h}{2}[L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)] + \beta v_h(y_\circ + hF(y_\circ, u))\right\}$$

Hamilton-Jacobi-Bellman (Part 3)

- Discrete HJB equation: for all $y_\circ \in \mathbb{R}^\ell$ and $\beta = e^{-\mu h}$

$$v_h(y_\circ) = \inf_{u \in U} \left\{ \tfrac{h}{2}[L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)] + \beta v_h(y_\circ + hF(y_\circ, u)) \right\}$$

- Define

$$S_h(y_\circ) = \operatorname*{argmin}_{u \in U} \left\{ \tfrac{h}{2}[L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)] + \beta v(y_\circ + hF(y_\circ, u)) \right\}$$

- Optimal feedback: $u_j^* = S_h(y_j^*)$, i.e.,

$$\begin{aligned} v_h(y_\circ) &= J_h(u_h^*; y_\circ) \\ y_{j+1}^* &= y_j^* + hF(y_j^*, S_h(y_j^*)) \text{ for } j \geq 0, \quad y_0^* = y_\circ \end{aligned}$$

- But: (discrete) HJB is difficult task for $\ell \geq 9$...

Numerical startegy (Part 1)

- Optimal control of evolution problems:

$$\min \mathrm{J}(\mathrm{y},\mathrm{u}) \quad \text{s.t.} \quad \dot{\mathrm{y}}(t) = \mathrm{F}(\mathrm{y}(t),\mathrm{u}(t)) \text{ for } t>0,\ \mathrm{y}(0)=\mathrm{y}_\circ,\ \mathrm{u}\in\mathcal{U}$$

- Galerkin approximation with $\psi_1,\dots,\psi_\ell$:

$$\min J(y,u) \quad \text{s.t.} \quad \dot{y}(t) = F(y(t),u(t)) \text{ for } t>0,\ y(0)=y_\circ,\ u\in U$$

- Trapezoidal sum for J and Euler's method
- Discrete HJB equation: $\beta = e^{-\mu h}$, for all $y_\circ \in \mathbb{R}^\ell$

$$v_h(y_\circ) = \inf_{u\in U} \Big\{ \frac{h}{2}\big(L(y_\circ,u)+\beta L(y_\circ+hF(y_\circ,u),u)\big)+\beta v_h(y_\circ+hF(y_\circ,u))\Big\}$$

Numerical startegy (Part 2)

- Utilize $\Upsilon_h = [a_1, b_1] \times \ldots \times [a_\ell, b_\ell] \subset \mathbb{R}^\ell$
- Rectlinear partition of Υ_h with vortices $y_j \in \mathbb{R}^\ell$
- Compute piecewise ℓ-linear $v_h^k : \Upsilon_h \to \mathbb{R}$ with

$$v_h^k(y_j) = \inf_{u\in U}\Big\{ \frac{h}{2}\big(L(y_\circ, u) + \beta L(y_\circ + hF(y_\circ, u), u)\big) + \beta v_h^k(y_\circ + hF(y_\circ, u))\Big\}$$

- Fixed point method with
  - nested iteration ($h = 0.2,\ h/4,\ h/16$)
  - parallelization

Boundary control of Burgers' equation

Consider

$$\min J(y,u) = \frac{1}{2}\int_0^\infty \Big(\int_\Omega |y(t,x)|^2 \,\mathrm{d}x + \beta \,|u(t)|^2 \Big) e^{-\mu t} \,\mathrm{d}t,$$

subject to the Burgers equation

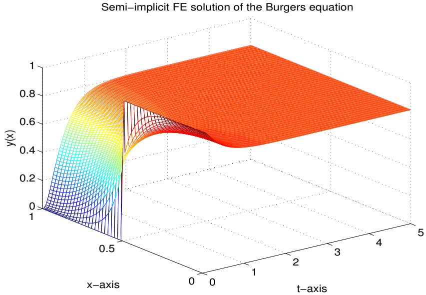
$$\begin{aligned} y_t - \nu y_{xx} + y y_x &= 0 && \text{in } Q \\ \nu y_x(\cdot,0) &= u && \text{in } (0,\infty) \\ \nu y_x(\cdot,1) &= 0 && \text{in } (0,\infty) \\ y(0,\cdot) &= y_\circ && \text{in } \Omega \end{aligned}$$

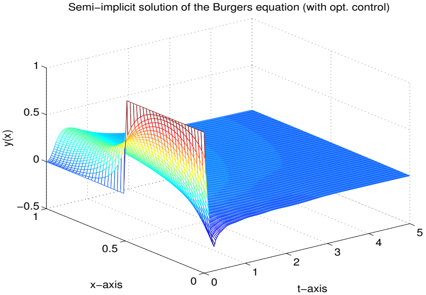
and control constraints

$$\mathcal{U} = \big\{ u \in L^2_{loc}(0,\infty) \;:\; u(t) \in U = [0,1] \text{ for } t \in (0,\infty) \big\}$$

Numerical Results (Part 1)

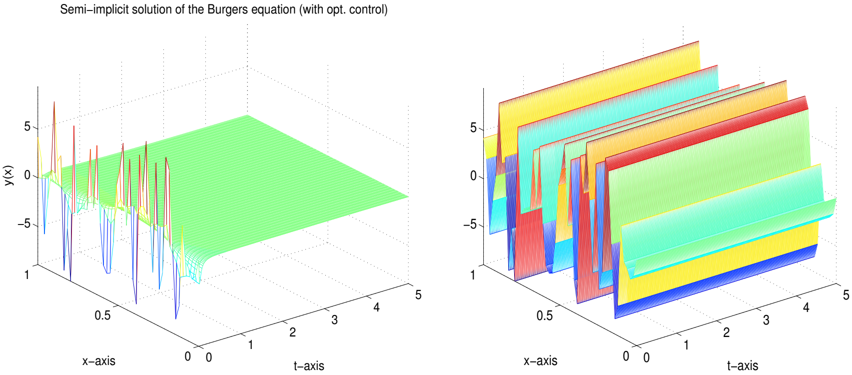
Discretization: $\ell = 4$ PODs; grid size for $\Upsilon \subset \mathbb{R}^\ell$: $24 \times 16 \times 4 \times 4$





	closed-loop (POD)	open-loop (FE)
uncontrolled cost	0.1258	0.1258
controlled cost	0.0370	0.0353
value function	0.0372	

Numerical Results (Part 2)



Closed-loop (left) and open-loop (right) solution for perturbed $y_\circ$

References

- ...

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- Leibfritz & V.: Numerical feedback controller design for PDE systems using model reduction: techniques and case studies, Proceedings, Santa Fe, USA, 2004
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