Universität Konstanz Sommersemester 2013
Fachbereich Mathematik und Statistik
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Optimierung

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 4

Deadline for hand-in: 2013/06/10 at lecture

Optimization with boundary constraints.
So far we looked for (local) minimizer $x^* \in \mathbb{R}^n$ of a sufficiently smooth and real valued function $f : \mathbb{R}^n \to \mathbb{R}$ in an open set $\Omega \subseteq \mathbb{R}^n$:

$$x^* = \underset{x\in\Omega}{\operatorname{argmin}} f(x).$$

By differential calculus, we immediately received as a first order necessary condition:

$$f(x^*) \leq f(x) \text{ for all } x \in B_\epsilon(x^*) \qquad \Longrightarrow \qquad \forall x \in \Omega : \langle \nabla f(x^*), x\rangle = 0.$$

If Ω is closed, e.g.,

$$\Omega = \prod_{i=1}^{n}[a_i, b_i] = \{x \in \mathbb{R}^n \mid \forall i = 1, ..., n : \; a_i \leq x_i \leq b_i \,, \; a_i, b_i \in \mathbb{R}\},$$

the situation turns out to be slightly more complicated: if a (local) minimizer is located on the boundary, the gradient condition is not longer a necessary criterion. We will focus on that in the next exercise.

Exercise 10
Let $f \in \mathcal{C}^2(\Omega^\circ, \mathbb{R})$, Ω as defined above. Notice that $\nabla f : \Omega^\circ \to \mathbb{R}^n$ can be expanded on the boundary of Ω since $f \in \mathcal{C}^2$ implies that ∇f is Lipschitz continuous on Ω°. Further, let $x^* \in \Omega$ be a local minimizer of f, i.e.

$$\exists \; \epsilon > 0 : \; \forall x \in B_\epsilon(x^*) \cap \Omega : \; f(x^*) \leq f(x).$$

Show that the following modified first order condition holds:

$$\forall x \in \Omega : \langle \nabla f(x^*), x - x^*\rangle \geq 0.$$

Any x^* that fulfills this condition is called *stationary point* of f.

Exercise 11

For $f : \mathbb{R}^n \to \mathbb{R}$ let L be the Lipschitz constant of the gradient ∇f. The *canonical projection* of $x \in \mathbb{R}^n$ on the closed set $\Omega = \prod_{i=1}^n [a_i, b_i]$ is given by $P : \mathbb{R}^n \to \Omega$,

$$\big(P(x)\big)_i := \begin{cases} a_i & \text{if } x_i \leq a_i \\ x_i & \text{if } x_i \in (a_i, b_i) \\ b_i & \text{if } x_i \geq b_i \end{cases} .$$

Further we define

$$x(\lambda) := P(x - \lambda \nabla f(x)).$$

Prove that the following ***modified Armijo condition*** holds for all $\lambda \in \left(0, \dfrac{2(1-\alpha)}{L}\right]$:

$$f(x(\lambda)) - f(x) \;\leq\; -\frac{\alpha}{\lambda}\|x - x(\lambda)\|^2_{\mathbb{R}^n}.$$

Hints: The following ansatz with the fundamental theorem of calculus may be helpful:

$$f(x(\lambda)) - f(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f\Big(x - t\big(x - x(\lambda)\big)\Big)\mathrm{d}t.$$

You can use the following formula without proof:

$$\big\langle x - x(\lambda),\; x(\lambda) - x + \lambda \nabla f(x)\big\rangle \;\geq\; 0.$$

Exercise 12 (4 Points)

Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ and the following algorithm:

Algorithm 1 (Projected Gradient)

while *termination criterion* is not fulfilled **do**
  while *modified Armijo condition* is not fulfilled **do**
    set $\lambda = \frac{\lambda}{2}$;
  end while
  set $x = x(\lambda)$;
end while

Let $(x_n)_{n \in \mathbb{N}}$ be now an iteration sequence created by this algorithm.

1. Show that $(f(x_n))_{n \in \mathbb{N}}$ converges.
2. Show that $(x_n)_{n \in \mathbb{N}}$ has at least one convergent subsequence.
3. Show that all accumulation points of $(x_n)_{n \in \mathbb{N}}$ are stationary points of f.
4. Show that x^* is a stationary point of f if and only if $x^* = P(x^* - \lambda \nabla f(x^*))$ holds.