Universität Konstanz Sommersemester 2013
Fachbereich Mathematik und Statistik
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# Optimierung

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

## Sheet 5

**Deadline for hand-in: 2013/06/24 at lecture**

**Exercise 13**
Let $f, g : \mathbb{R} \to \mathbb{R}$ be four-times differentiable functions where $g$ is an approximation of $f$:

$$g(x) = f(x) + r(x)$$

with some error function $r : \mathbb{R} \to \mathbb{R}$ and

$$|r(x)| \leq \epsilon$$

for some known $\epsilon > 0$.
Determine the numerical first derivative $D_c^1(g, h)$ of $g$ by central differences, where $h$ denotes the grid spacing. Compute the error arising in the numerical approximation of the first derivative. Use this result to show that the error arising in the numerical approximation of the second derivative by using again central differences is of order $O(\epsilon^{\frac{4}{9}})$.

**Exercise 14**
Let $f \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ and consider a uniform $n$-dimensional mesh with spacing $h$ and $e_i \in \mathbb{R}^n$, $i = 1, ..., n$, the canonical $i$th basis vector. Verify the (2nd order forward difference) formula

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x) = \frac{f(x + he_i + he_j) - f(x + he_i) - f(x + he_j) + f(x)}{h^2} + \mathrm{O}(h),$$

$i, j = 1, ..., n$, for an approximation of the Hessian matrix.

**Exercise 15** (4 Points)
Let $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R})$ be a quadratic function of the form

$$f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle + \gamma,$$

$Q \in \mathbb{R}^{n \times n}$ symmetric and positive definite, $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. Let $x_0 \in \mathbb{R}^n$ and $H$ be a symmetric, positive definite matrix.

Define $\tilde{f}(x) := f(H^{-\frac{1}{2}}x)$ and $\tilde{x}_0 = H^{\frac{1}{2}}x_0$. Let $(\tilde{x}_k)_{k\in\mathbb{N}}$ be a sequence generated by the steepest descent method,

$$\tilde{x}_{k+1} = \tilde{x}_k + \tilde{t}_k\tilde{d}_k \quad \text{with} \quad \tilde{d}_k = -\nabla\tilde{f}(\tilde{x}_k)$$

and $\tilde{t}_k = t(\tilde{d}_k)$ the optimal stepsize choice as determined in Exercise 5.

Let $(x_k)_{k\in\mathbb{N}}$ be generated by the gradient-like method with preconditioner $H$,

$$x_{k+1} = x_k + t_k d_k \quad \text{with} \quad d_k = H^{-1}(-\nabla f(x_k))$$

and $t_k = t(d_k)$ the optimal stepsize choice as determined in Exercise 5.

Show (by induction) that the two optimization methods are equivalent, i.e., for all $k \in \mathbb{N}$ it holds:

$$x_k = H^{-\frac{1}{2}}\tilde{x}_k.$$