Universität Konstanz
Fachbereich Mathematik und Statistik
Prof. Dr. Stefan Volkwein
Sabrina Rogg

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Optimierung

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 4

Deadline for hand-in: 30.05.2016 at lecture

Exercise 10 (Newton's method for finding roots) (3 Points)
Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = x^3 - 2x + 2 \quad \text{and} \quad g(x) = \sin(x).$$

(1) We apply Newton's method to the function f with starting point $x_0 = 0$. Show that the Newton iteration has two accumulation points which are both not roots of f.

(2) Find a starting point x_0 such that the Newton iteration for the funtion g is of form $x_k = x_0 + k\pi$, $k \in \mathbb{N}_{>0}$.

(3) Generate suitable plots (Matlab) for illustrating the non-convergence in (1) and (2).

Optimal control problem

We consider the following optimization problem:

$$\min J(y, u) \quad \text{subject to} \quad Ay = Bu \tag{1}$$

with $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$ invertible and $B \in \mathbb{R}^{n\times m}$.

So far, we have only considered optimization problems where the unknowns play similar roles. Since the matrix A^{-1} exists this is different here. For each $u \in \mathbb{R}^m$ ("arbitrarily chosen") there exists an associated solution $y \in \mathbb{R}^n$ to $Ay = Bu$ by

$$y = A^{-1}Bu. \tag{2}$$

This is why we call u the control and y the associated state.
We can introduce the reduced cost functional $f : \mathbb{R}^m \to \mathbb{R}$ as

$$f(u) := J(A^{-1}Bu, u), \tag{3}$$

which depends only on the control variable. This gives the unconstrained optimization problem

$$\min_{u\in\mathbb{R}^m} f(u). \tag{4}$$

In the following we consider the quadratic cost functional

$$J(y,u) = \frac{1}{2}\|y - y_d\|_2^2 + \frac{\lambda}{2}\|u\|_2^2, \tag{5}$$

where $\lambda > 0$.

Exercise 11 (2 Points)

- Write down the reduced cost functional for (5) and compute the reduced gradient $\nabla f(u)$.
- We derive an optimality system from the necessary condition $\nabla f(u^*) = 0$. It consists of three equations containing no inverse matrices.
  Let $u^* \in \mathbb{R}^m$ with $\nabla f(u^*) = 0$. Perform the following steps:
  1. The first equation is $Ay^* = Bu^*$. Reinsert the state y^* into the formula for $\nabla f(u^*)$.
  2. We introduce an additional variable: The adjoint state $p^* \in \mathbb{R}^n$ solving
     $$A^T p^* = (y^* - y_d). \tag{6}$$
     This is the second equation of the optimality system. Now, insert p^* into the formula for $\nabla f(u^*)$.
  3. Derive the third equation by solving $\nabla f(u^*) = 0$.

Exercise 12 (Lagrange formalisme) (2 Points)
Define the Lagrange function by

$$\begin{aligned} &L : \mathbb{R}^{2n+m} \to \mathbb{R} \\ &L(y,u,p) = J(y,u) - \langle Ay - Bu, p\rangle_2. \end{aligned} \tag{7}$$

Show that the optimality system from Exercise 11 is equivalent to

$$\nabla L(y^*, u^*, p^*) = 0, \tag{8}$$

which is the necessary optimality condition known from Analysis II.