Universität Konstanz
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Optimierung

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 5

Deadline for hand-in: 13.06.2016 at lecture

Exercise 13 (3 Points)

We consider the problem of finding the point on the parabola $y = \frac{1}{5}(u-1)^2$ which is closest to the point $(y, u) = (2, 1)^\top$. This can be formulated as

$$\min f(y, u) = (y-2)^2 + (u-1)^2 \quad \text{subject to} \quad 5y = (u-1)^2. \tag{1}$$

1. Introduce the Lagrange function $L : \mathbb{R}^3 \to \mathbb{R}$ for (1) and determine all critical points $(\tilde{y}, \tilde{u}, \tilde{p}) \in \mathbb{R}^3$ satisfying
$$\nabla L(\tilde{y}, \tilde{u}, \tilde{p}) = 0.$$

2. Eliminate the variable u from the cost functional by directly inserting the constraint. Show that the solutions of this reduced problem can not be solutions of the original one.

3. Which condition must be added in 2. so that the problems are equivalent?

Exercise 14 (Damped Newton) (2 Points)

Consider to use the classical Newton method for finding the root of $g(x) = \arctan(x)$. Let $x_k = 5$. Compute the next iteration point x_{k+1} according to the Newton algorithm

$$\begin{aligned} g'(x_k)\Delta x_k &= -g(x_k); \\ x_{k+1} &= x_k + \Delta x_k. \end{aligned}$$

The damped Newton algorithm is defined as

$$\begin{aligned} g'(x_k)\Delta x_k &= -g(x_k); \\ x_{k+1} &= x_k + \alpha_k \Delta x_k, \end{aligned}$$

with $\alpha_k \in (0, 1]$ computed by a line search procedure such that

$$|g(x_k + \alpha_k \Delta x_k)| \leq (1 - \delta\alpha_k)|g(x_k)|, \quad \delta \in (0, 1).$$

Let $\delta = 0.5$. We start from $\alpha = 1$ in the line search procedure and divide it by 2 every time the condition is not satisfied. Compute the point x_{k+1} using the damped Newton algorithm and compare it to the one returned by the classical Newton method. What do you conclude?

Exercise 15 (3 Points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be the quadratic function

$$f(x) = \frac{1}{2}\langle x, Qx\rangle + \langle c, x\rangle + \gamma,$$

with $Q \in \mathbb{R}^{n\times n}$ symmetric and positive definite, $c \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$. Let $x^0 \in \mathbb{R}^n$ and H be a symmetric positive definite matrix.

Define $\tilde{f}(x) := f(H^{-\frac{1}{2}}x)$ and $\tilde{x}^0 := H^{\frac{1}{2}}x^0$. Let $(\tilde{x}^k)_{k\in\mathbb{N}}$ be a sequence generated by the steepest descent method,

$$\tilde{x}^{k+1} = \tilde{x}_k + \tilde{t}_k\tilde{d}^k \quad \text{with} \quad \tilde{d}^k = -\nabla\tilde{f}(\tilde{x}_k)$$

and $\tilde{t}_k$ the optimal stepsize from Exercise 4 (for $\tilde{f}$).

Let $(x^k)_{k\in\mathbb{N}}$ be generated by the gradient-like method with preconditioner H,

$$x^{k+1} = x^k + t_k d^k \quad \text{with} \quad d^k = H^{-1}(-\nabla f(x^k))$$

and t_k the optimal stepsize from Exercise 4.

Show (by induction) that the two optimization methods are equivalent, i.e., for all $k \in \mathbb{N}$ it holds:

$$x^k = H^{-\frac{1}{2}}\tilde{x}^k.$$