Universität Konstanz
Fachbereich Mathematik und Statistik
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Wintersemester 16/17

Numerische Verfahren der restringierten Optimierung

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 5

Deadline for hand-in: 24.01.2017 at lecture

Note: Program 3 is already available online with submission on Tuesday, February 7, 10:00!

Exercise 12 (2 Points)

Let $\bar{x} \in \mathbb{R}^n$ be given, and let x^* be the solution of the projection problem

$$\min \|x - \bar{x}\|^2 \quad \text{subject to} \quad l \le x \le u.$$

For simplicity, assume that $-\infty < l_i < u_i < \infty$ for all $i = 1, 2, \ldots, n$. Show that the solution of this problem coincides with the projection formula given by

$$P(x, l, u)_i = \begin{cases} l_i & \text{if} \quad x_i < l_i, \\ x_i & \text{if} \quad x_i \in \ [l_i, u_i], \\ u_i & \text{if} \quad x_i > u_i, \end{cases}$$

that is, show that $x^* = P(\bar{x}, l, u)$.

Exercise 13 (2 Points)

Consider the quadratic optimization problem given by

$$\min \ f(x) := \frac{1}{2} x^\top Q x + x^\top d + c,$$

where $Q \in \mathbb{R}^{n \times n}$ is symmetric and positive definite. Let x^* be the minimizer of f and define the energy norm as $\|x\|_Q := (x^\top Q x)^{1/2}$. Show that the following equality holds:

$$f(x) = \frac{1}{2} \|x - x^*\|_Q^2 + f(x^*).$$

Exercise 14 (2 Points)

Consider the nonlinear optimization problem

$$\min f(x) \quad \text{subject to } e(x) = 0, \ g(x) \le 0, \tag{1}$$

for which the Lagrangian function is given by

$$L(x, \lambda, \mu) = f(x) + \lambda^\top e(x) + \mu^\top g(x) \quad \text{for } (x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p_+.$$

The *dual problem* to (1) is defined by

$$\sup_{\lambda \in \mathbb{R}^m, \mu \in \mathbb{R}^p_+} d(\lambda, \mu), \tag{2}$$

where $d(\lambda, \mu) := \inf_{x \in \mathbb{R}^n} L(x, \lambda, \mu)$ denotes the dual objective function. In this context we refer to the original problem (1) as the *primal problem.*

1. Show that the following *weak duality* result holds: For any $\tilde{x}$ feasible for (1) and any $(\tilde{\lambda}, \tilde{\mu}) \in \mathbb{R}^m \times \mathbb{R}^p_+$, we have

   $$d(\tilde{\lambda}, \tilde{\mu}) \leq f(\tilde{x}).$$

   It says that the optimal value of the *dual problem* gives a lower bound on the optimal value of the *primal problem.*

2. Derive the *dual problem* to the linear programming problem

   $$\min c^\top x \quad \text{subject to } Ax = b,\, x \geq 0,$$

   with $A \in \mathbb{R}^{m \times n}, c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.