

# Exercises for Theory and Numerics of Partial Differential Equations

<http://www.math.uni-konstanz.de/numerik/personen/beermann/en/teaching>

## Sheet 11

**Deadline: Thursday, 02/02, 5pm**

**Exercise 10.1** (Matlab)

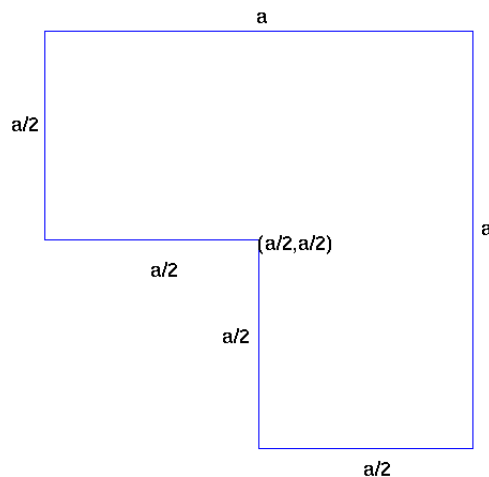
(10 points)

Please follow the *programming guidelines* that can be download under the above url.

Using MATLAB *PDE Toolbox*, solve the following Parabolic Problem:

$$\begin{cases} y_t(t, x) - \Delta y(t, x) = f(x) & \text{for all } x \in \Omega \\ \eta \frac{\partial y(t, x)}{\partial n} + \alpha y(t, x) = 0 & \text{for all } x \in \partial\Omega \\ y(0, x) = y_0(x) \end{cases} \quad (1)$$

which depends on  $\alpha, \eta \in \mathbb{R}$ ,  $\eta \neq 0$ , and where  $t \in [0, 1]$ ,  $f : \Omega \rightarrow \mathbb{R}$  is a continuous function and  $\Omega \subset \mathbb{R}^2$  is given by the interior of the blue line, that depends on the parameter  $a > 0$  in  $\mathbb{R}$  as shown in figure:



As in the previous Sheet, declare  $a$  as global parameter in your main script and make it available in each function. In order to solve the problem follow these steps:

1. **Geometry Implementation:** Write a function `geometryFunction.m` to describe the geometry of  $\Omega$  by using a suitable analytical boundary representation. Especially focus on the various way that this function will be called by the PDE toolbox (0,1,2 inputs, bs scalar or a vector,...) Then use the command `pdegplot('geometryFunction')` to test your results.

The following two points should be solved in a script.  
**Do not use point-and-click for these!**

2. **PDE specification:** Specify the PDE coefficients in (1) and generate a mesh with maximum element size 0.05. Visualize the mesh.

3. **PDE solving:** Solve the problem for different choices of the parameters  $\eta$  and  $\alpha^1$  and for the following choices of  $f(x)$  and  $y_0(x)$ :

- (a)  $f(x) = 0$ ,  $y_0(x) = x_1 + x_2$  with  $x = (x_1, x_2) \in \Omega$ ;
- (b)  $f(x) = a$ ,  $y_0(x) = x_1 + x_2$ ;
- (c)  $f(x) = \frac{1}{4}a^2 - (x_1 - \frac{1}{2}a)^2 - (x_2 - \frac{1}{2}a)^2$ ,  $y_0(x) = a$ ;

In particular, try the choice  $(\eta, \alpha) = (1, 10^7)$ , which kind of boundary condition does it imitate? Write a function `solve_parabolic_problem`, which solves the problem by assembling the Finite Element matrices with the command `assembleFEMatrices` and use *implicit Euler*<sup>2</sup> scheme in time to get the solution<sup>3</sup>  $y(t_s, \cdot)$  at every time step  $t_s$ .<sup>4</sup> The function should also plot the time evolution of the solution  $y(t, x)$  with  $t \in [0, 1]$  and  $x \in \Omega$ .

Write a thorough report documenting how the variation of  $\eta$ ,  $\alpha$  and  $N_t$ , the number of time steps used for the time discretization, affects the solution.

### Exercise 10.2 (Theory)

(10 points)

Let  $T > 0$  be a final time and  $V, H$  Hilbert spaces such that  $V \hookrightarrow H = H' \hookrightarrow V'$  is a *Gelfand triple*<sup>5</sup>. Let further  $A \in L(V, V')$  and  $f \in L^\infty(0, T, V')$  as well as  $y_0 \in H$ . If  $A$  is *coercive*, meaning that there are  $\alpha > 0$ ,  $\beta \geq 0$  with

$$\langle A\varphi, \varphi \rangle_{V' \times V} \geq \alpha \|\varphi\|_V^2 - \beta \|\varphi\|_H^2 \quad \forall \varphi \in V$$

then the following is known as an *abstract parabolic evolution equation*:

$$\begin{aligned} y_t(t) + Ay(t) &= f(t) && \text{in } V' \text{ for almost all } t \in (0, T) \\ y(0) &= y_0 && \text{in } H \end{aligned} \tag{PB}$$

It can be shown that a unique solution of (PB) exists with

$$y \in W(0, T) := L^2(0, T; V) \cap H^1(0, T; V') \hookrightarrow C([0, T]; H)$$

where the last embedding is a well-known property of the space  $W(0, T)$ . Your task is to prove the following Theorem which estimates the energy of the solution  $y$  against the energy of the initial data  $f$  and  $y_0$ :

**Theorem 1.** For all solutions  $y \in W(0, T)$  of (PB), it holds

$$\|y(T)\|_H^2 + \alpha \|y\|_{L^2(0, T; V)}^2 \leq e^{2\beta T} \left( \|y_0\|_H^2 + \frac{1}{\alpha} \|f\|_{L^2(0, T; V')}^2 \right)$$

Do this by the following steps:

1. Derive an estimate (1) for the term  $\|y(t)\|_H^2$ . In order to do this, you may use the fact that  $\langle y_t(t), y(t) \rangle_{V' \times V} = \frac{1}{2} \frac{d}{dt} \|y(t)\|_H^2$ . Young's inequality is also helpful.
2. Use Gronwall's Lemma<sup>6</sup> to derive from 1. an estimate (2) for the term  $\|y(t)\|_H^2$ .
3. Integrate your original estimate (1) over  $(0, T)$  and use your estimate (2) to derive (PB).

<sup>1</sup>MATLAB let you set only the  $\alpha$  coefficient in a simple way, so we suggest to divide the boundary equation by  $\eta$  to obtain the new equation  $\frac{\partial y(t, x)}{\partial n} + \frac{\alpha}{\eta} y(t, x) = 0$ . Notice, also, that as in the previous sheet for some combination of parameter, the solution is really 'ugly'.

<sup>2</sup>Note well: this scheme is stable for every choice of time discretization.

<sup>3</sup>Use the `\` command to solve the linear systems.

<sup>4</sup>There are several ways to solve (1) in MATLAB, but, in order to let you learn something, we recommend this procedure. For `assembleFEMatrices` look at this link: <https://de.mathworks.com/help/pde/ug/assemblefematrices.html>

<sup>5</sup>This means that  $V$  is densely embedded in  $H$  and  $H'$  is densely embedded in  $V'$ .  $H$  is identified with  $H'$  by the Riesz isomorphism.

<sup>6</sup>See <https://www.math.uni-bielefeld.de/~rkruse/files/gronwall.pdf>