Universität Konstanz *SS 10*
Fachbereich Mathematik und Statistik
S. Volkwein, O. Lass, R. Mancini

Übungen zu **Numerik gewöhnlicher Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 3 **Submission: 20.05.2010, 12:00 o'clock, Box 13**

Exercise 7 (Homework) (4 Points)

Consider the following positive-definite matrix

$$A = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{(n-1)\times(n-1)}.$$

Verify that the eigenvalues of A are

$$\lambda_k = 4\sin^2\left(\frac{k\pi}{2n}\right) \qquad k = 1,\dots,n-1$$

with associated eigenvectors

$$v_k = \left(\sin\left(\frac{k\pi}{n}\right), \sin\left(\frac{2k\pi}{n}\right), \dots, \sin\left(\frac{(n-1)k\pi}{n}\right)\right)^T \in \mathbb{R}^{n-1}.$$

Exercise 8

Write the *heat equation*

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}, \quad t > 0,\ x \in (0,\ell)$$

as a system of $(n-1)$ ODEs and show that the obtained linear system is stiff.

Hint: A linear system $y' = Ay$ with A an $(n-1)\times(n-1)$ matrix is called *stiff* if

$$\mathrm{Re}(\lambda_i) < 0 \quad \text{and} \quad \max_{i,j} \frac{|\lambda_i|}{|\lambda_j|} \gg 1$$

for the eigenvalues $\lambda_i \in \mathbb{C}$, $i = 1,\dots,n-1$ of A.

Exercise 9

Formulate the *trapezoidal method* for the linear system obtained in Exercise 8. How high are the computational costs in each iteration?

Program 2 (10 Points)

Consider the linear system obtained in Exercise 8, i.e.

$$y' = Ay$$

with $\kappa = 1$, $\ell = 1$ and the initial condition $y_0 = (\Phi(x_0), \ldots, \Phi(x_{n-1}))^T$ with $\Phi(x) = \sin(\pi x)$. Choose $n = 60$ for the numerical experiments.

1. Implement the *midpoint rule*

$$\begin{aligned} y_{k+\frac{1}{2}} &= y_k + \frac{h}{2} f(t_k, y_k), \\ y_{k+1} &= y_k + hf\left(t_k + \frac{h}{2}, y_{k+\frac{1}{2}}\right) \end{aligned}$$

   on the ODE system obtained in Exercise 8. Plot and compare the results obtained with different stepsizes h. What do you observe? How small a stepsize do you have to choose to get *good* results? Show what happens if h is chosen wrong.

   Hint: The midpoint rule is only *stable* if $|h\lambda_{n-1}(A)| < 2$, where $\lambda_{n-1}(A)$ is the largest eigenvalue of A.

2. Implement the *trapezoidal method* on the linear system of Exercise 8. Hence, compare the results obtained with different stepsizes h.