Universität Konstanz
Fachbereich Mathematik und Statistik
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# Optimization
# Exercises 3

**✓ Exercise 9** (5 Points)

(1) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := (x_1 + x_2^2)^2$$

in the point $x_0 = (1, 0)$.

Show that $d := (-1, 1)$ is a direction of descent and find all minimal points of the problem

$$\min_{\alpha > 0} f(x_0 + \alpha d).$$

(2) Consider the function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad (x_1, x_2) \mapsto f(x_1, x_2) := 3x_1^4 - 4x_1^2 x_2 + x_2^2.$$

Prove that $x_0 := (0, 0)$ is a stationary point of $f$. Show that $f$, restricted on any line through $x_0$, has a strict local minimum in $x_0$. Is $x_0$ a local minimizer of $f$?

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Let $\mathcal{H}$ denote a $\mathbb{K}$-Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

**✓ Exercise 10** (5 Points)

(1) Let $b \in \mathcal{H}$ and $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$, the space of all linear, continuous maps on $\mathcal{H}$.

Show that $x_0 \in \mathcal{H}$ is a minimal point of

$$\varphi : \mathcal{H} \to \mathbb{R}, \qquad x \mapsto \varphi(x) := ||Ax - b||_{\mathcal{H}}$$

if and only if the Gaussian normal equation holds:

$$A^* A x_0 = A^* b.$$

Hereby, $A^*$ denotes the adjoint operator to $A$, i.e. the following implicitely given operator $A \in \mathcal{L}_b(\mathcal{H}, \mathcal{H})$:

$$\forall x, y \in \mathcal{H} : \langle Ax, y \rangle_{\mathcal{H}} = \langle x, A^* y \rangle_{\mathcal{H}}.$$

(2) Use this characterization to solve the following linear regression problem:

Find parameters $x_1, x_2 \in \mathbb{R}$ such that the corresponding regression line

$$\gamma_x : \mathbb{R} \to \mathbb{R}, \qquad t \mapsto \gamma_x(t) := x_1 + x_2 t$$

approximates the measuring points

| $t_i$ | 1975 | 1980 | 1985 | 1990 | 1995 |
|---|---|---|---|---|---|
| $\gamma_i$ | 30 | 35 | 38 | 42 | 44 |

optimally, i.e.

$$(x_1, x_2) = \underset{(y_1, y_2)}{\arg\min} \sum_{i=1}^{5} (\gamma_i - \gamma_y(t_i))^2.$$

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**Exercise 11**

Let $x \in \mathcal{H}$ and $F$ a convex, nonempty, closed subset of $\mathcal{H}$.

Show that there is a unique $y \in \mathcal{H}$ such that

$$||x - y||_{\mathcal{H}} = \operatorname{dist}(x, F).$$

Hereby, dist denotes the distance function $\operatorname{dist}(y_0, Y) := \inf_{y \in Y} ||y_0 - y||_{\mathcal{H}}$.

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**Exercise 12**

Let $F$ a nonempty, closed, convex subset of $\mathcal{H}$ and $x_0 \in \mathcal{H}$.

Show that for all $x \in \mathcal{H}$ the following holds:

$$||x_0 - x||_{\mathcal{H}} = \operatorname{dist}(x_0, F) \qquad \Longleftrightarrow \qquad \forall y \in F : \operatorname{Re}\langle x_0 - x, y - x \rangle_{\mathcal{H}} \leq 0.$$

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**Deadline:** Monday, 16$^{\text{th}}$ May, 8:30 am