Universität Konstanz *WS 10/11*
Fachbereich Mathematik und Statistik
S. Volkwein, O. Lass, R. Mancini

Übungen zu **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 2 **Submission: 20.12.2010, 11:00 o'clock, Box 18**

Exercise 4 (4 Points)

We consider the discretization of the partial differential equation

$$(Lu)(x,y) = f(x,y) \quad \text{for all } (x,y) \in \Omega = (0,1) \times (0,1), \tag{1a}$$

where

$$(Lu)(x,y) = -\Delta u(x,y) + a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y)$$

for all (x,y) on the unit square $\Omega = (0,1) \times (0,1)$ with homogeneous Dirichlet boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0 \quad \text{for } x,y \in [0,1]. \tag{1b}$$

For general coefficients a, b, c, $f \in C(\bar{\Omega})$ and $c \geq 0$ in Ω the operator L is not self-adjoint and its discretization is not symmetric.

Discretize (1a) by a five-point centered difference scheme with n^2 points and mesh width $h = 1/(n+1)$. Moreover, the partial derivatives u_x and u_y are also discretized by utilizing centered difference schemes. The unknowns are denoted by

$$u_{ij} \approx u(x_i, y_j),$$

where $x_i = ih$ for $i = 1, \ldots, n$. Compute the coefficient matrix $A \in \mathbb{R}^{n^2 \times n^2}$ and the right-hand side $b \in \mathbb{R}^{n^2}$ so that the discretization of (1) can be formulated as a linear system of the form

$$Au = b. \tag{2}$$

When is the matrix A a L_0-matrix?

Hint: For needed definitions see the lecture notes on *Numerik gewöhnlicher Differentialgleichungen* by Prof. S. Volkwein.

Exercise 5 (4 Points)

In (2) the computation of Au can be done matrix-free. Write a pseudo code that realizes the product Au.

Exercise 6 (4 Points)

Let A be a block-tridiagonal matrix of the form

$$A = \begin{pmatrix} A_1 & C_1 & 0 & \dots & 0 \\ B_2 & A_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{n-1} & A_{n-1} & C_{n-1} \\ 0 & \dots & 0 & B_n & A_n \end{pmatrix}, \tag{3}$$

where the A_l's ($1 \le l \le n$) are quadratic matrices of the size m_l. Further $B_l \in \mathbb{R}^{m_l \times m_{l-1}}$ for $l = 2, \dots, n$ and $C_l \in \mathbb{R}^{m_l \times m_{l+1}}$ for $l = 1, \dots, n-1$ hold.

- Derive an algorithm, which realizes the factorization

$$A = \begin{pmatrix} D_1 & 0 & 0 & \dots & 0 \\ B_2 & D_2 & & & \\ & \ddots & \ddots & & \\ & & B_{n-1} & D_{n-1} & 0 \\ 0 & \dots & 0 & B_n & D_n \end{pmatrix} \begin{pmatrix} E_1 & F_1 & 0 & \dots & 0 \\ 0 & E_2 & F_2 & & \\ & & \ddots & \ddots & \\ & & & E_{n-1} & F_{n-1} \\ 0 & \dots & 0 & 0 & E_n \end{pmatrix} =: LU, \tag{4}$$

  where $E_l \in \mathbb{R}^{m_l \times m_l}$ denote identity matrices.

  Hint: If it is necessary, suppose the invertibility of certain matrices.

- Assume that the matrices

$$A^{(l)} = \begin{pmatrix} A_1 & C_1 & 0 & \dots & 0 \\ B_2 & A_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & B_{l-1} & A_{l-1} & C_{l-1} \\ 0 & \dots & 0 & B_l & A_l \end{pmatrix}, l = 1, \dots, n-1, \tag{5}$$

  are non-singular. Show that D_l^{-1} exist for $l = 1, \dots, n-1$.

  Hint: Prove the assertion by induction and use $\det(AB) = \det(A)\det(B)$.