*Universität Konstanz* *WS 10/11*
*Fachbereich Mathematik und Statistik*
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# Übungen zu **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

**Sheet 4** **Submission: 17.01.2011, 11:00 o'clock, Box 18**

**Exercise 10** (4 Points)

Let $\Omega = (a, b) \subset \mathbb{R}$. Decompose $\Omega$ into subdomains $\Omega_i$ using the mesh $\{x_i\}_{i=0}^N$:

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b.$$

Set $\Omega_i = (x_{i-1}, x_i)$ for $i = 1, \ldots, N$. Furthermore, we introduce the local step size notation $h_i := x_i - x_{i-1}$ for $i = 1, \ldots, N$. Compute the piecewise linear and globally continuous nodal basis functions. Further draw graphs of the basis function and compute the derivatives. Repeat this for piecewise quadratic and globally continuous nodal basis functions.

**Exercise 11** (4 Points)

Let us consider the boundary value problem

$$-(p(x)u')' + q(x)u = f(x), \qquad x \in \Omega \tag{1}$$

$$u(0) = 0,\ u(1) = 0, \tag{2}$$

where $\Omega = (0, 1)$, $p \in C(\overline{\Omega})$, $q \in C(\overline{\Omega})$, $f \in L^2(\Omega)$ with $p(x) \geq \overline{c} > 0$ and $q(x) \leq 0$ for all $x \in \overline{\Omega}$.

1. State the weak formulation of the problem.

2. Using piecewise linear and globally continuous nodal basis functions on a uniform mesh of size $h = 1/N$, $N \geq 2$, write down the finite element approximation to this problem if $p$ and $q$ are assumed to be constant. What do you observe when looking at the obtained system compared to a finite difference discretization?

3. Replace (2) by
$$u(0) = 0 \quad \text{and} \quad u'(1) = 0.$$
Write down the finite element approximation to this new problem (i.e., repeat steps 1 and 2). What changes compared to the Dirichlet problem?

**Exercise 12** (4 Points)

Given $A \in \mathbb{R}^{N\times N}$ a symmetric and positive definite matrix. The vectors $d_0, d_1, \ldots, d_{N-1} \in \mathbb{R}^N$ are called $A$-orthogonal if $d_i^T A d_j = 0$ for $i \neq j$, $0 \leq i, j \leq N-1$ and $d_i^T A d_i \neq 0$ for $0 \leq i \leq N-1$.

a) Verify that $d_0, d_1, \ldots, d_{N-1}$ are a basis of $\mathbb{R}^N$.

b) Given the equation $Ax = b$ show that the solution $x = A^{-1}b$ can be written with the basis $d_0, \ldots, d_{N-1}$ as:

$$x = \sum_{k=0}^{N-1} \alpha_k d_k \quad \text{with} \quad \alpha_k = \frac{d_k^T A x}{d_k^T A d_k}, \; k = 0, \ldots, N-1.$$

c) Show that for every $x_0 \in \mathbb{R}^N$ the sequence

$$x_{k+1} = x_k + \alpha_k d_k \quad \text{with} \quad \alpha_k = \frac{-d_k^T (A x_k - b)}{d_k^T A d_k}$$

for $k \geq 0$ gives the solution $x_N = A^{-1}b$ after at most $N$ steps.