Universität Konstanz *WS 11/12*
Fachbereich Mathematik und Statistik
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ÜBUNGEN ZU **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 1

Deadline: 2011/12/16 at lecture

Note:

- Please write **each exercise** on a **separate sheet**!
- Remember to write **name, Matrikel-Nummer, sheet number, exercise number** and your **group** on **each sheet**!

Exercise 1 (4 Points)

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a connected and bounded domain with C^1 boundary $\partial\Omega$. Let $D_\psi = \{u \in C(\bar{\Omega});\ u = \psi \text{ on } \partial\Omega\}$. We consider the differential operator L:

$$L : C^2(\Omega) \cap D_0 \longrightarrow C(\Omega) \quad , \quad Lu = -\Delta u\,.$$

Show:

1. $\langle Lu, v\rangle = \langle u, Lv\rangle$ for $u, v \in C^2(\Omega) \cap D_0$,
2. L is positive definite.
3. Consider a square domain, e.g. $\Omega = [a, b]^2$. Can the results above be applied in this situation? Give a reference and cite an appropriate theorem.

The scalar product of two function u and v on Ω is defined by

$$\langle u, v\rangle = \int_\Omega uv\,\mathrm{d}x\,.$$

Exercise 2 (4 Points)

Consider the linear problem

$$\begin{cases} -u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \text{ for all } x \in (0,1), \\ u(0) = \alpha, u(1) = \beta, \end{cases} \tag{1}$$

where b, c, f are continuous functions on $[0,1] \subset \mathbb{R}$ with $c > 0$ and α, β are real scalars.

1. Discretize (6) using the step size $h = \frac{1}{N}$, $N \in \mathbb{N}$, and central/symmetric difference approximations for $u''(x)$ und $u'(x)$. Present the linear system.

2. What are the conditions that the coefficient matrix of the obtained linear system is strictly diagonal dominant? What changes if $u'(x)$ is approximated as follows:

$$u'(x) \approx \begin{cases} \dfrac{u(x+h)-u(x)}{h} & \text{if } b(x) < 0, \\ \dfrac{u(x)-u(x-h)}{h} & \text{if } b(x) \geq 0. \end{cases} \tag{2}$$

3. Name an application where the (strictly) diagonal dominance is important.

Remark: The approximation (5) is called *upwind scheme*. This kind of discretization is often used in the context of convection dominated equations.

Hint: Recall that a matrix $A = ((a_{ij})) \in \mathbb{R}^{N\times N}$ is *strictly row diagonal dominant* if

$$\sum_{j=1, j\neq i}^{N} |a_{ij}| < |a_{ii}| \quad \text{for all } i \in \{1, \dots, N\}.$$

The matrix A is called *row diagonal dominant* if

$$\sum_{j=1, j\neq i}^{N} |a_{ij}| \leq |a_{ii}| \quad \text{for all } i \in \{1, \dots, N\}$$

holds (see Lecture Notes Numerik I, page 16).

Exercise 3 (4 Points)

We consider the following elliptic problem in *divergence form*:

$$\big(a(x)v'(x)\big)' = d(x) \quad \text{for } x \in \Omega = (0,1), \tag{3a}$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \geq \underline{a} > 0$ for all $x \in \Omega$ and $d \in C(\Omega)$. We state Dirichlet boundary conditions

$$v(0) = \alpha, \quad v(1) = \beta \tag{3b}$$

with $\alpha, \beta \in \mathbb{R}$.

The goal is to obtain a symmetric coefficient matrix when discritizing (6) by finite differences.

1. Discretize the interval Ω using the equidistant grid $x_i = ih$, $0 \leq i \leq n+1$, with the mesh size $h = 1/(n+1)$, $n \in \mathbb{N}$. Approximate the outer derivative $(aw)'$ at the grid point x_i by central difference quotients using the **intermediate** grid points $x_{i\pm 1/2} = x_i \pm h/2$, $i = 1, \dots, n$. Write $a_i = a(x_i)$, $d_i = d(x_i)$ for $i = 0, \dots, n+1$ and $a_{i\pm 1/2} = a(x_i \pm h/2)$ for $i = 1, \dots, n$. Use the formula:

$$f'(x_i) \approx \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{h}$$

2. Replace w by v' now and discretize v' in the symmetric difference for $(av')'$ by central differences. What are the equations that you derive for the approximation of (6)? Compose the coefficient matrix explicitly.