Universität Konstanz *WS 11/12*
Fachbereich Mathematik und Statistik
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ÜBUNGEN ZU **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 3 **Submission: 10.01.2011, 10:30 o'clock, Box 18**

Exercise 7 (4 Points)

Let $A \in \mathbb{R}^{M^2 \times M^2}$ be the matrix obtained by the classical finite difference method for solving the boundary value problem $-\Delta u = g$ in $\Omega = (0,1) \times (0,1)$ and $u = \gamma$ on $\partial\Omega$ with stepsize $h = \frac{1}{M+1}$. Show that the vectors $u^{kl} \in \mathbb{R}^{M^2}$,

$$(u^{kl})_{ij} \;=\; \sin\left(\frac{ik\pi}{M+1}\right)\sin\left(\frac{jl\pi}{M+1}\right), \quad 1 \le i, j \le M$$

are the eigenvectors of A. What are the corresponding eigenvalues λ_{kl}?

Exercise 8 (4 Points)

Consider the problem

$$\begin{aligned} -\Delta v &= \lambda v \text{ in } \Omega, \\ v|_{\partial\Omega} &= 0, \end{aligned} \qquad (1)$$

with $\Omega \subset \mathbb{R}^2$ a bounded domain with piecewise smooth boundary $\partial\Omega$.
A solution $v \in C^2(\Omega) \cap C(\bar{\Omega})$, $v \neq 0$ is called an eigenfunction to the eigenvalue λ.

a) Show that all eigenvalues λ of (1) are positive.

b) Let v_1, v_2 be eigenfunctions to the corresponding eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$. Show that v_1, v_2 are orthogonal with respect to the scalar product

$$\langle u, w \rangle = \int_\Omega u(x)w(x)dx.$$

c) Let $\Omega = (0,1)^2$. Show that the eigenvalues of (1) are given by $\lambda_{kl} = \pi^2(k^2 + l^2)$. Compare the corresponding eigenfunctions with those of Exercise 7.

d) Show that the difference between the eigenvalues in Exercise 7 and the corresponding eigenvalues in Exercise 8 is of the order $O(h^2)$.

Exercise 9 (4 Points)

Consider the elliptic differential equation with Neumann condition on the boundary

$$\Delta u(x,y) = f(x,y) \quad \text{in } \Omega \tag{2}$$

$$\frac{\partial u}{\partial \vec{n}} = g(x,y) \quad \text{in } \Gamma = \partial\Omega \tag{3}$$

where Ω is a rectangle domain $(0,a)\times(0,b)$. To simplify matters we consider a uniformly equidistant grid, i.e., we choose grid points (ih, jh) for $i = 0,1,...,M$ and $j = 0,1,...,N$ such that $Mh = a$ and $Nh = b$.

We have to distinguish between four different types of grid points:

- **inner points**, e.g. $\bullet^{\cdot,I}_{(\cdot,\cdot)}$
- **boundary points**, e.g. $\bullet^{\cdot,B}_{(\cdot,\cdot)}$
- **corner points**, e.g. $\bullet^{\cdot,C}_{(\cdot,\cdot)}$
- **ghost points**[1], e.g. $\circ^{g}_{(\cdot,\cdot)}$

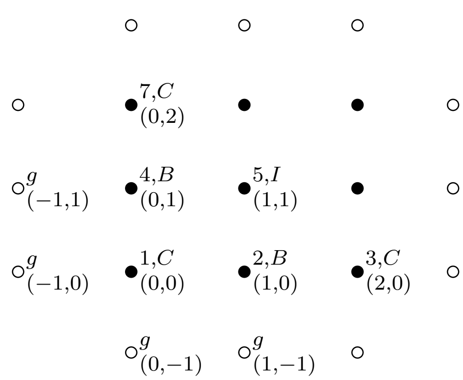


Fig. 1

The subscript indicates the index pairs $(\cdot,\cdot)$ of the point, while the superscript contains the "point number" and the indicator for the "point type".

1. Formulate difference equations for the problem by using the **five-point stencil**

$$\Delta u(x,y) \approx \frac{u(x-h,y) + u(x+h,y) + u(x,y-h) + u(x,y+h) - 4u(x,y)}{h^2}$$

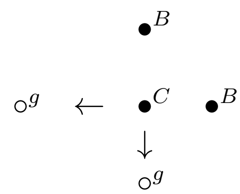
for all grid points (ih, jh), $i = 0,1,...,M$ and $j = 0,1,...,N$. Here the ghost points will be needed! Note the tacit assumption that the right-hand side f is also defined

[1] Ghost points are no "real" grid points but they appear in the formulation of the finite differences. *Hint*: They can be "compensated" by reformulating the information obtained by the finite differences scheme with respect to the grid points.

on Γ. For this formulation, approximate the Neumann condition $\frac{\partial u}{\partial \vec{n}}$ on boundary points by central differences:

$$
\begin{aligned}
u'(x,y) &\approx \frac{u(x+h,y)-u(x-h,y)}{2h} \quad (x\text{-direction})\,, \\
u'(x,y) &\approx \frac{u(x,y+h)-u(x,y-h)}{2h} \quad (y\text{-direction})\,.
\end{aligned}
$$

At the corner points, where $\vec{n}$ is undefined, approximate the "normal derivative" by the average of the two derivatives along the two outer normals to the sides meeting at the corner (use also central differences):



In the formulation for the boundary points as well as in the formulation for the corner points the ghost points will appear (see the *Hint* in the footnote).

2. Formulate explicitly the system matrix for $M = N = 2$ and $g \equiv 0$ (see Fig. 1).

3. Assume again $g = 0$. Show that solutions to the problem can not be unique. Furthermore, show that this matches with the fact of the non-invertibility of the discretization matrix.

Merry Christmas and a happy new year!