Universität Konstanz *WS 11/12*
Fachbereich Mathematik und Statistik
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Übungen zu **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 5 **Deadline: 27.01.2012, 10:10 o'clock before the Lecture**

Exercise 13 (Quadratic ansatz functions) (4 Points)

Let $\Omega = (a,b) \subseteq \mathbb{R}$ and $x_0, ..., x_{n+1} \in \bar{\Omega}$ with $x_0 = a,\ x_{n+1} = b$, $x_i < x_{i+1}$.

1. Define piecewise quadratic ansatz functions ϕ_i, $i = 1, ..., n$, and $\phi_{i+\frac{1}{2}}$, $i = 0, ..., n$, in $\mathcal{C}^0(\bar{\Omega})$ such that

$$\begin{aligned} \phi_i(x_j) &= \delta_{ij} \qquad \& \qquad \phi_i(x_{j+\frac{1}{2}}) = 0, \\ \phi_{i+\frac{1}{2}}(x_j) &= 0 \qquad \& \qquad \phi_{i+\frac{1}{2}}(x_{j+\frac{1}{2}}) = \delta_{ij}. \end{aligned}$$

   Hint: A quadratic function is determined uniquely by its values on three interpolation points.

2. Calculate the derivatives $\phi_i, \phi_{i+\frac{1}{2}}$.

3. Draw $\phi_i, \phi_{i+\frac{1}{2}}, \phi_i', \phi_{i+\frac{1}{2}}'$ for one fixed i in different colours in one **large** plot. Don't forget to label your axes.

4. Consider the ansatz

$$y(x) = \sum_{i=1}^{n} \alpha_i \phi_i(x) + \sum_{i=0}^{n} \alpha_{i+\frac{1}{2}} \phi_{i+\frac{1}{2}}(x)$$

   for arbitrary coefficients $\alpha_i, \alpha_{i+\frac{1}{2}} \in \mathbb{R}$. Explain why y in general is just of $\mathcal{H}^1$-regularity (you don't need to prove this).

Exercise 14 (Weak derivatives) (4 Points)

Let $\Omega = (-1, 1)$.

1. Let $u \in \mathcal{L}^2(\Omega)$ Define: $v \in \mathcal{L}^2(\Omega)$ is the weak derivative of u.

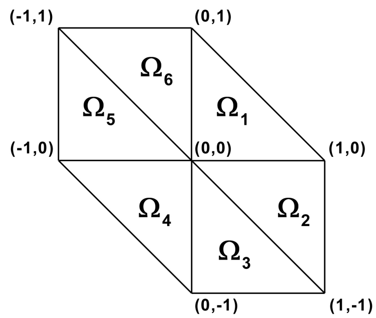
2. $u(x) = \text{abs}(x)$. Show that the weak derivative $v = u' \in \mathcal{L}^2(\Omega)$ of u is given by

$$u'(x) = \begin{cases} -1 & x \in [-1, 0] \\ 1 & x \in [0, 1] \end{cases} \ .$$

3. Prove that u' is not weakly differentiable in $\mathcal{L}^2(\Omega)$.

<u>Exercise 15</u> (Ansatz functions in two dimensions) (4 Points)

Let $\Omega = (-1,1) \times (-1,1)$. Consider the following triangulation of Ω:



1. Define the function $S : \bar{\Omega} \to \mathbb{R}$ describing the canonical simplex on Ω, i.e. $S_i = S|_{\bar{\Omega}_i}$ is a plain with $S_i(0,0) = 1$ and $S_i(x,y) = 0$ for all other grid points $(x,y) \in \bar{\Omega}$.

2. Draw the graph of S.

3. Assume that Ω is triangulized by n^2 inner grid points x_i, y_j now. Let $f \in \mathcal{C}^0(\bar{\Omega})$, $f_{ij} = f(x_i, y_j)$ and
$$F(x,y) = \sum_{i,j=1}^{n} f_{ij}\phi_{ij}(x,y)$$
where ϕ_{ij} is the simplex with centre (x_i, y_j).

   Describe the graph of F.

4. Is F a good approximation for f?