Universität Konstanz *WS 11/12*
Fachbereich Mathematik und Statistik
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ÜBUNGEN ZU **Theorie und Numerik partieller Differentialgleichungen**

http://www.math.uni-konstanz.de/numerik/personen/volkwein/teaching/

Sheet 6 **Deadline: 03.02.2012, 10:10 o'clock before the Lecture**

Exercise 16 (Mass and stiffness matrices) (4 Points)

1. Let y the solution to a 1D elliptic or parabolic problem on $\Omega = (0, 1)$ with Dirichlet boundary conditions $y(0) = 0$ and $y(1) = 1$. Compute the mass matrix Φ and the stiffness matrix Ψ for the polynomial basis functions $\phi_i(x) = x^i$, $i = 1, ..., n$. What disadvantage do you see compared to the ansatz with piecewise linear finite elements?

2. Compute the condition numbers of the matrices Ψ and $\frac{1}{k}\Phi + \Psi$ for the polynomial and the piecewise linear ansatz with $n = 15$ and $k = 0.01$. Notice that the slope of the hat functions is $h = \frac{1}{n}$ here since a half-hat is needed for the right boundary. What do you observe?

3. What tells us the condition number of a matrix?

Hint: The condition number of a symmetric and positive definite matrix S is given by the quotient of the maximal and the minimal eigenvalue. You may use the MATLAB commands `eig` or `eigs`, respectively, to compute the eigenvalues of the matrices.

Exercise 17 (The `pdetool` of MATLAB) (4 Points)

Consider the elliptic problem

$$
\begin{aligned}
-\Delta u(x,y) &= 0 && (x,y) \in (0,2) \times (-1,1), \\
u(x,y) &= x(2-x) && (x,y) \in (0,2) \times \{-1,1\} \\
u(x,y) &= (y+1)(y-1) && (x,y) \in \{0,2\} \times (-1,1).
\end{aligned} \tag{1}
$$

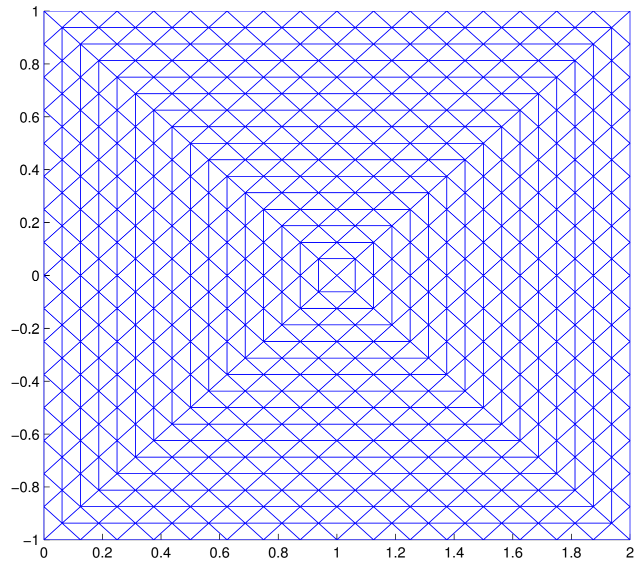
Solve (1) by using the graphical user interface `pdetool` of the Partial Differential Equation Toolbox in MATLAB.

Follow the steps **Draw**, **Boundary**, **PDE**, **Mesh** and **Solve**. Finally plot the solution.

Hint: A short and good summary on the use of the `pdetool` can be found in the web by Prof. HEINRICH VOSS with the name *Eine sehr kurze Einführung in die Partial Differential Equation Toolbox von* MATLAB. The hyperlink is

`www.tu-harburg.de/rzt/tuinfo/software/numsoft/matlab/pde/pdetool.ps`

Try to find the correct parameters to generate the symmetric grid



and plot the solution in this case, too.

Remark: Your submission for this exercise shall just include the 3D-plots.

Exercise 18 (A nonlinear problem) (4 Points)

Consider the nonlinear elliptic equation

$$\begin{aligned} -\Delta u &= f(u) \quad \text{on } \Omega \\ u &= 0 \qquad\ \text{on } \partial\Omega \end{aligned} \tag{2}$$

on a (bounded) square $\Omega \subseteq \mathbb{R}^2$ and $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that f' is bounded.

Let $x = (x_0, ..., x_{n+1})$ and $y = (y_0, ..., y_{n+1})$ be an equidistant discretization of $\bar{\Omega}$ and A the five-point-stencil discretization matrix for $-\Delta$. Then (2) reads as

$$AU = F(U) \quad \text{on } \{(x_i, y_j) \mid i = 1, ..., n,\ j = 1, ..., m\} \tag{3}$$

where $U = (U_{ij}) \in \mathbb{R}^{n^2}$, $U_{ij} \approx u(x_i, y_j)$, and $F(U) = (f(U_{ij})) \in \mathbb{R}^{n^2}$.

1. Assume that (3) admits a unique solution $U \in \mathbb{R}^{n^2}$. Define the formal iteration sequence $(U^k)_{k\in\mathbb{N}}$ of the Newton method for some starting guess $U^1 \in \mathbb{R}^{n^2}$. Present the matrices arising in this formulation explicitely.

2. Show that there is an $\varepsilon > 0$ such that the iteration sequence is well-defined for all stepsizes $h < \varepsilon$ and all initializations U^1.

3. What convergence rate do you get for $(U^k)_{k\in\mathbb{N}}$ if U^1 is chosen sufficiently close to U?