POSITIVITY, VALUATIONS, AND QUADRATIC FORMS

KONSTANZ, 1-6 OCTOBER 2009

Schedule

Thursday, 1 October

9:15 – 9:30	*Address of welcome*
9:30 – 10:30	M. Laurent: *Optimization over polynomials with sums of squares and moment matrices*
10:30 – 11:00	*— Coffee break —*
11:00 – 11:30	H. Bosse: *Non-negative polynomials that are not sums of squares*
11:40 – 12:10	T. Theobald: *Valuations and tropical bases*
12:20 – 12:50	J. Povh: *Ncsostools: a computer algebra system for symbolic and numerical computation with nc polynomials*
13:00	*— Lunch at restaurant "Arche" —*
15:00 – 16:00	D. Hoffmann: *Something old and something new about u*
16:00 – 16:30	*— Coffee break —*
16:30 – 17:00	A. Smith: *Fast positivity testing using Bernstein expansion*
17:15 – 17:45	J. Cimpric: *A method for computing lowest eigenvalues of symmetric polynomial differential operators by semidefinite programming*

Friday, 2 October

9:15 – 10:15	J. Koenigsmann: *Fields with the Galois group of* $\mathbb{Q}$
10:15 – 10:45	*— Coffee break —*
10:45 – 11:15	J. Demeyer: *Hilbert's tenth problem for function fields over valued fields*
11:30 – 12:00	S. Wagner: *A decision problem for real multivariate polynomials*
12:15 – 12:45	F. Acquistapace: *Positive semidefinite analytic functions*
13:00	*— Lunch —*
15:00 – 16:00	R. Baeza: *Generalized Hermite constants*
16:00 – 16:30	*— Coffee break —*
16:30 – 17:30	K. Schmüdgen: *Positivity, sums of squares and positivstellensätze for* $*$*-algebras*
18:00	*Reception in honour of Alexander Prestel*

Saturday, 3 October

9:15 – 10:15	J.-P. Tignol: *Valuations on central simple algebras*
10:15 – 10:35	— *Coffee break* —
10:35 – 11:05	R. Flatley: *Trace forms of symbol algebras*
11:15 – 12:15	L. van den Dries: *Immediate extensions of H-fields*
12:25 – 12:55	M. Dickmann: *Faithfully quadratic rings*
13:00	— *Lunch at restaurant "Arche"* —
19:30	*Conference dinner at restaurant "Konzil"*

Monday, 5 October

9:15 – 10:15	P. Roquette: *Arf invariants of quadratic forms in historical perspective*
10:15 – 10:45	— *Coffee break* —
10:45 – 11:15	M. Knebusch: *Semirings with bounds*
11:30 – 12:00	N. Lavi: *Ganzstellensatz for open sets in real closed fields*
12:15 – 12:45	H. Nguyen: *Polynomials non-negative on non-compact subsets of the plane*
13:00	— *Lunch* —
15:00 – 16:00	W.-D. Geyer: *Patching*
16:00 – 16:30	— *Coffee break* —
16:30 – 17:00	R. Quarez: *Effective tridiagonal determinantal representation for univariate polynomials and real roots counting*
17:15 – 17:45	P. Gładki: *Certain quotient spaces of spaces of orderings and their inverse limits*

Tuesday, 6 October

9:15 – 10:15	J. Hartmann: *Local global principles for quadratic forms*
10:15 – 10:45	— *Coffee break* —
10:45 – 11:15	A. Pfister: *An elementary and constructive proof of Hilbert's theorm on ternary quartics*
11:30 – 12:00	Y. Savchuk: *Positivstellensätze for some algebras of matrices*
12:15 – 12:45	M. Karoubi: *Periodicity in hermitian K-theory*
13:00	— *Lunch* —
15:00 – 16:00	I. Klep: *Free real algebraic geometry*
16:00 – 16:30	— *Coffee break* —
16:30 – 17:00	M. Thomas: *Parameterization and the rational points of definable sets*
17:15 – 17:45	T. Unger: *Torsion in Witt groups and sums of hermitian squares*

Talks and abstracts

Francesca Acquistapace (Pisa)
Positive semidefinite analytic functions
Friday, 12:15

Abstract. A positive semidefinite analytic function f on $\mathbb{R}^n$ was known to be a sum of squares of meromorphic functions only when the set $f^{-1}(0)$ was compact (Ruiz 1985, also Jaworski), or discrete (Bochnak, Kucharz, Shiota 1981). I will present some recent improvements of those results.

Ricardo Baeza (Talca)
Generalized Hermite constants
Friday, 15:00

Abstract. To any number field K and positive integer one can associate a constant which in the case of $K = \mathbb{Q}$ coincides with the classical Hermite constant. We will survey some results concerning these constants, and how one can compute some of them for quadratic number fields.

Hartwig Bosse (Frankfurt)
Non-negative polynomials that are not sums of squares
Thursday, 11:00

Jaka Cimpric (Ljubljana)
A method for computing lowest eigenvalues of symmetric polynomial differential operators by semidefinite programming
Thursday, 17:15

Jeroen Demeyer (Ghent)
Hilbert's tenth problem for function fields over valued fields
Friday, 10:45

Abstract. A diophantine equation is a polynomial equation with integer coefficients in an arbitrary number of variables. Hilbert's Tenth Problem was the following question: find an algorithm which, given a diophantine equation, tells whether or not it has a solution over the integers. It was shown in 1970 by Y. Matiyasevich, building on earlier work by M. Davis, H. Putnam and J. Robinson, that this question has a negative answer: such an algorithm does not exist. In other words: diophantine equations are undecidable.

Many authors have generalized this undecidability to other rings and fields. Two important open cases are $\mathbb{Q}$ and $\mathbb{C}(x)$. In this talk, I will show that equations over $\mathbb{C}((t))(x)$ are undecidable. This is a special case of a more general theorem which works for function fields over a valued field in equal characteristic zero with some extra conditions. Quadratic forms play an important role in the proof.

Max Dickmann (Paris)
Faithfully quadratic rings
Saturday, 12:25

Ronan Flatley (Dublin)
Trace forms of symbol algebras
Saturday, 10:35

Abstract. Let S be a symbol algebra. The trace form of S is computed and it is shown how this form can be used to determine whether S is a division algebra or not.

Wulf-Dieter Geyer (Erlangen)
Patching
Monday, 15:00
Abstract. In the last decades there have been developed at least four different but related versions of patching: The analytic patching, the formal patching, the rigid patching and the algebraic patching. This talk specialises patching to the construction of galois covers, i.e. to the Inverse Galois Problem, especially in the analytic and in the algebraic case.

Paweł Gładki (Santa Barabara)
Certain quotient spaces of spaces of orderings and their inverse limits
Tuesday, 11:30

Julia Hartmann (Aachen)
Local global principles for quadratic forms
Tuesday, 9:15
Abstract. This talk is concerned with local global principles for isotropy of quadratic forms over function fields of curves over complete discretely valued fields. We also obtain a short exact sequence for the corresponding Witt groups. The results are obtained using patching methods.
(Joint work with David Harbater and Daniel Krashen)

Detlev Hoffmann (Nottingham)
Something old and something new about u
Thursday, 15:00
Abstract. The u-invariant resp. the Hasse number $\widetilde{u}$ of a field is the smallest nonnegative integer n such that every torsion resp. totally indefinite quadratic form of dimension $> n$ is isotropic provided such an n exists, and it is defined to be infinity otherwise. In the case of a nonreal field, every form is a torsion form, and every form is totally indefinite since this condition is vacuous in the absence of orderings, so both invariants coincide and are equal to the supremum of the dimensions of anisotropic forms over that field. For real fields, one always has $u \leq \widetilde{u}$ but equality need not hold. We present some old and new results on the relation between these invariants and on conditions that guarantee their finiteness.

Max Karoubi (Paris)
Periodicity in hermitian K-theory
Tuesday, 12:15
Abstract. It is known since few years, by the work of Voevodsky and Rost, that higher algebraic K-theory of a commutative ring A with suitable finite coefficients is periodic above the étale dimension of A. In this lecture, we prove that any ring A with periodic K-theory above a certain range has also a periodic K-theory for quadratic forms above this range. This lecture is prepared for a non specialist audience. Therefore, all the basic definitions will be given.

Igor Klep (Ljubljana)
Free real algebraic geometry
Tuesday, 15:00
Abstract. In this talk we will sketch a few of the developments in the emerging area of real algebraic geometry over a free $*$-algebra, in particular on "*noncommutative inequalities*".
Let $n \in \mathbb{N}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $\mathbb{K}\langle \underline{X}, \underline{X}^* \rangle$ denote the free $*$-algebra on $\underline{X}$, that is, the set of all $\mathbb{K}$-linear combinations of words in $\underline{X} := (X_1, \ldots, X_n)$ and $\underline{X}^* := (X_1^*, \ldots, X_n^*)$. Such elements are called *NC polynomials*. For $f \in \mathbb{K}\langle \underline{X}, \underline{X}^* \rangle$ and an n-tuple $\underline{A} = (A_1, \ldots, A_n)$ of matrices of the same size we consider the evaluation $f(\underline{A}) = f(A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)$.

Two types of results will be discussed: one parallels classical real algebra with sums of squares representations (Positivstellensätze) for positive NC polynomials, and the other has a different flavor focusing on NC semialgebraic sets and NC mappings between them. These sets are defined as positivity domains of NC polynomials and include NC balls, and solution sets of NC linear matrix inequalities (LMIs). Here is a sample:

Theorem 1 (Helton). *Suppose $f \in \mathbb{K}\langle \underline{X}, \underline{X}^* \rangle$ satisfies $f(\underline{A})$ is positive semidefinite for all n-tuples $\underline{A}$ of matrices of the same size. Then f is a sum of hermitian squares.*

Theorem 2 (Helton, McCullough, K.). *Let h be an origin and boundary preserving NC mapping between NC balls $\mathbb{B}_n := \{\underline{B} = (B_1, \dots, B_n) \mid \|\underline{B}\| \leq 1\}$ and $\mathbb{B}_m$. Then h is linear and there is a unique isometry $U \in \mathbb{K}^{m \times n}$ satisfying $h = U\underline{X}$. In particular, if $m < n$ then no such NC mapping exist.*

Manfred Knebusch (Regensburg)
Semirings with bounds
Monday, 10;45

Abtract. We call a (commutative) semiring R a *semiring with upper bounds* (or ub-semiring for short) if the addition on R gives a partial ordering on R such that, for any two elements x, y of R, the sum $x + y$ is an upper bound of x and y. (It may be bigger than the maximum of x and y which perhaps does not exist). This new notion in semiring theory generalizes the notion of an upper bound group invented recently by Niels Schwartz.

Ub-semirings give a natural frame to study families of valuations on semirings, and thus take a natural place in real algebra. The semiring of all sums of squares in a field is a case in point.

An important further class of ub-semirings are the supertropical semirings invented by Zur Izhakian. They allow to refine the valuations on a semiring (in particular on a field) to "supervaluations" (joint work with Zur Izhakian and Louis Rowen).

Jochen Koenigsmann (Oxford)
Fields with the Galois group of $\mathbb{Q}$
Friday, 9:15

Abstract. Using valuation theory and quadratic forms, we show that a field F whose absolute Galois group $G_F = \mathrm{Gal}(F^{sep}/F)$ is isomorphic to $G_{\mathbb{Q}}$ shares many arithmetic properties with $\mathbb{Q}$. We will report on recent progress towards a classification of such fields and the impact on the birational *Section Conjecture* in Grothendieck's Anabelian Geometry.

Monique Laurent (Amsterdam)
Optimization over polynomials with sums of squares and moment matrices
Thursday, 9:30

Abstract. Polynomial optimization deals with the problem of minimizing a multivariate polynomial over a basic closed semi-algebraic set K defined by polynomial inequalities and equations. While polynomial time solvable when all polynomials are linear (via linear programming), the problem becomes hard in general as soon as it involves non-linear polynomials. Just adding the simple quadratic constraints $x_i^2 = x_i$ on the variables, already makes the problem NP-hard.

A natural approach is then to relax the problem and to consider easier to solve, convex relaxations. The basic idea, which goes back to work of Hilbert, is to relax non-negative polynomials by sums of squares of polynomials, a notion which can be tested efficiently (using semidefinite programming algorithms). On the dual side, one views points as (atomic) measures and one searches for efficient conditions characterizing sequences of moments of non-negative measures on the semi-algebraic set K. In this way hierarchies of efficient convex relaxations can be build.

We will review their various properties. In particular, convergence properties, that rely on real algebraic geometry representation results for positive polynomials, stopping criteria and extraction of global minimers, that rely on results from moment theory and commutative algebra. We will consider in particular unconstrained polynomial optimization problems and problems with a finite real variety.

Noa Lavi (Be'er Sheva)
Ganzstellensatz for open sets in real closed fields
Monday, 11:30
Abstract. For valued fields, instead of being positive or non-negative, one might consider the property of being integral. A rational function will be called integral-definite over a definable set S if it takes only values in the valuation ring (ganze element). One can search for an algebraic representation of such functions.
The aim of the talk will be showing a model theoretic framework for proving such ganzstellensatz theorem for open sets intersected by integrality set of n functions in real closed fields.

Ha Nguyen (Atlanta)
Polynomials non-negative on non-compact subsets of the plane
Monday, 12:15
Abstract. Recently, M. Marshall answered a long-standing question by showing that if $f(x,y) \in \mathbb{R}[x,y]$ is non-negative on the strip $[0,1] \times \mathbb{R}$, then f has a representation $f = \sigma_0 + \sigma_1(1-x)$, where $\sigma_0, \sigma_1 \in \mathbb{R}[x,y]$ are sums of squares. In this talk we present some generalizations of this result to other non-compact basic closed semialgebraic sets of $\mathbb{R}^2$ which are contained in the strip. We also give some negative results.

Albrecht Pfister (Mainz)
An elementary and constructive proof of Hilbert's theorm on ternary quartics
Tuesday, 10:45

Janez Povh (Ljubljana)
Ncsostools: a computer algebra system for symbolic and numerical computation with nc polynomials
Thursday, 12:20

Ronan Quarez (Rennes)
Effective tridiagonal determinantal representation for univariate polynomials and real roots counting
Monday, 16:30
Abstract. We show how Sturm and Sylvester algorithms, which both compute the number of real roots of a given univariate polynomial over the reals, lead to two tridiagonal determinantal representations that can be viewed as dual.

Peter Roquette (Heidelberg)
Arf invariants of quadratic forms in historical perspective
Monday, 9:15
Abstract. Cahit Arf, a mathematician from Turkey, studied in 1937/38 as a doctoral student with Helmut Hasse. In his thesis he proved the Arf part of what now is known as the Hasse-Arf theorem which is of importance in class field theory. During his stay in Göttingen he met Ernst Witt and there developed a friendship between the two, The topic of my lecture will be Arf's second paper where he transferred Witt's theory of quadratic forms to fields of characteristic 2. In the course of his investigation he discovered what is now known as the Arf invariant of a quadratic form. This is of relevance to local and global fields of characteristic 2, and has also applications in topology.
I will report on this from a historical perspective on the basis of letters, manuscripts and documents which are preserved at the Handschriftenabteilung of the Göttingen library. There are more than 60 letters between Hasse and Arf.
In later years Cahit Arf has become a leading figure in the mathematics scene of Turkey. The new 10 Lira note of Turkish currency carries the portrait of Arf together with the formula:

$$\mathrm{Arf}(q) = \sum_{i=1..n} q(a_i)q(b_i) \in \mathbb{Z}_2$$

for Arf invariants. As I will point out this formula does not really represent the scope of Arf"s discovery.

Yuriy Savchuk (Leipzig)
Positivstellensätze for some algebras of matrices
Monday, 17:15

Konrad Schmüdgen (Leipzig)
Positivity, sums of squares and positivstellensätze for $*$-algebras
Friday, 16:30

Abstract. This talk will be concerned with various notions and results on positivity of symmetric elements of noncommutative $*$-algebras. Let A be a (complex or real) unital $*$-algebra with involution $a \to a^*$ and let $\mathcal{A}_h$ be the set of symmetric elements $a = a^*$ of A. A subset $\mathcal{C}$ of $\mathcal{A}_h$ is called a quadratic module if $1 \in \mathcal{C}$, $\lambda_1 c_1 + \lambda_2 c_2 \in \mathcal{C}$ and $a^* c a \in \mathcal{C}$ for all $c_1, c_2, c \in \mathcal{C}, \lambda_1 1, \lambda_2 \in [0, \infty)$ and $a \in \mathcal{A}$. There are quadratic modules defined in algebraic terms (as finite weigthed sums of squares $a^* c a$ for all $a \in \mathcal{A}$ and c in some fixed subset of $\mathcal{A}_h$) and quadratic modules defined by Hilbert space representations (as those elements of $\mathcal{A}_h$ which are mapped into positive operators under some $*$-representations of A). Noncommutative Positivstellensätze deal with the interplay between these two classes of quadratic modules. Some new Positivstellensätze are presented for $*$-algebras of matrices over commutative or noncommutative $*$-algebras and for $*$-algebras of fractions.

Andrew Smith (Konstanz)
Fast positivity testing using Bernstein expansion
Thursday, 16:30

Abstract. It is known that the coefficients of the Bernstein expansion of a given multivariate polynomial over a specified box of interest tightly bound the range of the polynomial over the box. The problem of testing the positivity of such a polynomial over a box can thus be reduced to the problem of computing its Bernstein coefficients. The traditional approach, however, requires that all such coefficients are computed, and their number is often very large for polynomials with moderately-many variables. A more efficient method for the implicit representation and computation of Bernstein coefficients of multivariate polynomials is presented. The complexity becomes nearly linear with respect to the number of terms in the polynomial, instead of exponential with respect to the number of variables. The Bernstein enclosure can also be used to construct affine underestimating bound functions for polynomials which can be employed in a branch-and-bound framework for solving constrained global optimization problems.

Thorsten Theobald (Frankfurt)
Valuations and tropical bases
Thursday, 11:40

Abstract. Tropical geometry deals with the images of algebraic varieties under real valuations. The resulting tropical varieties are polyhedral cell complexes which preserve many properties of complex algebraic varieties.
Among the key concepts in tropical geometry is the one of a tropical basis, which is a basis $g_1, \ldots, g_m$ of an ideal I such that the intersection of the tropical hypersurfaces $T(g_i)$ coincides with the tropical variety $T(I)$. In this talk, we exhibit the concept of tropical bases, show how to construct short bases by means of regular projections and discuss these projection-based bases.
(Based on joint work with Kerstin Hept.)

Margaret Thomas (Oxford)
Parameterization and the rational points of definable sets
Tuesday, 16:30

Abstract. We provide some background to the problem of bounding the density of rational points

lying on transcendental sets, in the context of o-minimal expansions of the reals. In particular, we focus on strategies involving parameterization of definable sets - coverings by the images of definable functions with bounded derivatives. We shall look at some recent work towards a conjecture of Wilkie about the real exponential field and, time permitting, present some results about particular definable sets and whether they do - or do not - have certain kinds of parameterizations.

Jean-Pierre Tignol (Louvain-la-Neuve)
Valuations on central simple algebras
Saturday, 9:15
Abstract. Valuation theory plays a central role in the solution of various problems concerning finite-dimensional division algebras, such as the construction of noncrossed products and of counterexamples to the Kneser-Tits conjecture. However, relating valuations with Brauer-group properties is particularly difficult because valuations are defined only on division algebras and not on central simple algebras with zero divisors. This talk will present a more flexible tool recently developed in a joint work with Adrian Wadsworth, which applies to a broad spectrum of noncommutative situations. In particular, central simple algebras with anisotropic involution over Henselian fields are shown to carry a special kind of value function, which is an analogue of Schilling valuations on division algebras.

Thomas Unger (Dublin)
Torsion in Witt groups and sums of hermitian squares
Tuesday, 17:15
Abstract. Let F be a formally real field. It is well-known that the Witt ring $W(F)$ is torsion-free iff every sum of squares in F is again a square in F. For a central simple F-algebra A with involution σ of the first kind, consider the Witt group $W(A,\sigma)$. One can ask if $W(A,\sigma)$ is torsion-free iff every sum of hermitian squares is again a hermitian square. The answer is "yes" in certain simple situations, but "no" in general as I will demonstrate. This is joint work with Vincent Astier.

Sven Wagner (Konstanz)
A decision problem for real multivariate polynomials
Friday, 11:30
Abstract. Given any finite sequence $h = (h_1, \dots, h_s)$ of real polynomials in n variables $X_1, \dots, X_n$ with $W(h) = \{x \in \mathbb{R}^n \mid h_1(x) \geqslant 0, \dots, h_s(x) \geqslant 0\}$ non-empty and bounded, we are interested whether every real polynomial f which is strictly positive on $W(h)$ admits a representation

$$f = \sigma_0 + h_1\sigma_1 + \cdots + h_s\sigma_s$$

where each σ_j is a sum of squares of polynomials in $\mathbb{R}[X_1, \dots, X_n]$.
If $n = 1$, this always holds, but if $n \geqslant 2$, this is not true in general. For $n = 2$, Canto Cabral has given an effective decision procedure that decides whether h has this property. We have shown decidability for every choice of n, and in this talk we want to give a short overview of the proof.

Lou van den Dries (Urbana)
Immediate extensions of H-fields
Saturday, 11:15
Abstract. This is joint work with Aschenbrenner and van der Hoeven. H-fields are ordered differential fields where the ordering and derivation interact as in Hardy fields and as in the differential field of logarithmic-exponential series. To obtain a model theory of these valued differential fields we need to understand their immediate extensions. This summer we made progress in this direction, by showing that if K is a Liouville closed H-field, then the maximal immediate extension of the underlying valued field of K can be made into an H-field extension of K.