

Ganzstellensatz for open sets in real closed fields

Noa Lavi

joint work with Yoav Yaffe

Mathematics Department
Ben Gurion University of the Negev

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The problem

Let $\langle K, \nu, \leq_K \rangle$ be a real closed valued field, and let

$$S_{\vec{p}=\langle p_1, \dots, p_m \rangle, \vec{f}=\langle f_1, \dots, f_{m'} \rangle} = \{x \in K^n \mid p_i(x) > 0, \nu(f_j(x)) \geq 0\}$$

where $p_1, \dots, p_m \in K[\vec{x}]$ and $f_1, \dots, f_{m'} \in K(\vec{x})$.

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Q: Can one find a **good** algebraic characterization of all the rational functions which are **really** integral on S ?

A: **Yes!**

Structure of the talk

- What is "really"?
- How do we find the good? (by model theory)
- The generators of the good and order

OVF integrality

Definition

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A: Then it is not **really** integral on S .

Wrong answer!!!

Example

Let $f(x, y) = \frac{x^2}{x^2 + y^2}$, and let us assume that we have a structure of an ordered valued field on $L = K(x)$ extending K . Then $\nu_L(f) = \nu_L(x^2) - \nu_L(x^2 + y^2) \geq 0$. Therefore, f should be **really** integral on the entire K^2 plane, but it is undefined at $(0, 0)$.

OVF-Integrality

Definition

An extension $\tilde{\nu}$ of a valuation ν on a field K to a field extension L will be called an *OVF-valuation* if there exists an order \leq_L on L which extends the order of K such that $\tilde{\nu}$ and \leq_L form a structure of an ordered valued field on L .

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Definition

Let K be a real closed valued field, and let $b \in K^n$. An OVF-valuation $\tilde{\nu}$ on L is said to be given by evaluation near b (for short - valuation near b) if

- (i) $\tilde{\nu}(f) = \nu(f(b))$ if $f(b) \neq 0$
- (ii) $\tilde{\nu}(f) > \Gamma_K$ if $f(b) = 0$

Definition

given $f \in L$ and $b \in K^n$, we say that f is OVF-integral at b , if $\nu_b(f) \geq 0$ for every ν_b on L which is valuation near b . For $S \subseteq K^n$ we say that f is OVF-integral at S if and only if f is OVF-integral at b for every $b \in S$.

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Theorem

Let $f \in L$ and $b \in K^n$ such that $f(b)$ is defined. Then f is integral at b if and only if it is OVF-integral at b .

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Proof.

If $f(b) \neq 0$ then $\nu(f(b)) = \nu_b(f)$ whenever ν_b is a valuation near b . Hence by definition 5 the equivalence holds.

If $f(b) = 0$ then f is integral at b and for every ν_b valuation near b $\nu_b(f) > \gamma$ for every $\gamma \in \Gamma_K$. Therefore $\nu_b(f) > 0$. □

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Lemma

(Kochen) Let K be a valued field, L a field extension of K and A a sub-ring of L such that $A \cap K = O_K$. Then $\bigcap \sqrt[A_T]{A_T}$ is the intersection of all valuation subrings O_L of L such that $A \subseteq O_L$ and $O_L \cap K = O_K$

the model theoretic tools

Definition

A Theory T is said to be *model complete* if for every $\mathcal{M}, \mathcal{N} \models T$ such that $\mathcal{M} \subset \mathcal{N}$ and for every quantifier-free formula $\phi(\vec{x}, y_1, \dots, y_m)$ such that $y_1, \dots, y_m \in \mathcal{M}$ we have that if there exists $\vec{b} \in \mathcal{N}^n$ such that $T \models \phi(\vec{b}, y_1, \dots, y_m)$ then there exists $\vec{a} \in \mathcal{M}^n$ such that $T \models \phi(\vec{a}, y_1, \dots, y_m)$.

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Definition

Let T be a theory. We say that \tilde{T} is a *model companion* of T if \tilde{T} is model complete and for every $\mathcal{M} \models T$ There exists $\mathcal{N} \models \tilde{T}$ such that $\mathcal{M} \subseteq \mathcal{N}$ (i.e \mathcal{M} is a substructure of \mathcal{N}).

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(G. Cherlin and M. Dickmann) *The theory of real closed valued fields is the model companion of the theory of ordered valued fields.*

The important theorem above is the one that allows us now to define what a **good** A is going to be and prove that it is indeed such. That is, $\sqrt[\text{int}]{A_T}$ will be our final and good answer.

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Another well known usage of model completeness is the model theoretic proof of Hilbert's nullstellensatz which lies on the fact that the theory of algebraically closed field is model complete.

Let $\langle K, \nu, <_K \rangle \models RCVF$ and let $S \subseteq K^n$ be a nonempty set definable by quantifier free formula ϕ_S . Let A be an O_K -Algebra. We say that A is *good* if it has the following properties:

- **necessity** For every $f \in A$ we have that f is OVF-integral on S .

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- **necessity** For every $f \in A$ we have that f is OVF-integral on S .
- **sufficiency** Let $\tilde{\nu}$ be an OVF-valuation extending ν to L such that $A \subseteq O_{\tilde{\nu}}$. Then there exist an order $<_L$ extending $<_K$ such that $\mathcal{L} := \langle L, \tilde{\nu}, <_L \rangle$ is an ordered valued field and $\mathcal{L} \models \phi_S(X)$.

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- **extension** A contains a set I_{ord} such that for every $\tilde{\nu}$ extending ν to L , Then for every $f \in I_{ord}$ we have that $\tilde{\nu}(f) \geq 0$ if and only if $\tilde{\nu}$ is an OVF-valuation of ν to L .

Theorem

For every $h \in L$ we have that h is OVF-integral on S if and only if $h \in \sqrt[\text{int}]{A_T}$

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Outline of proof:

For every $h \notin \sqrt[\text{int}]{[A_T]}$ we use lemma of Kochen and the conditions above to construct a model extension of $\mathcal{K} := \langle K, \nu, \leq_K \rangle$ to a model of OVF \mathcal{L} where h does not belong to the valuation ring of L and ϕ_S holds. This lemma gives us a valuation $\tilde{\nu}$ extending ν to L such that $\tilde{\nu}(h) < 0$ and the extension and sufficiency conditions gives us a compatible linear order \leq_L extending \leq_K such that $\mathcal{L} \models \phi_S$. There exists a model of RCVF extending the previous such that the same holds. Hence, according to the model completeness of RCVF the same holds in \mathcal{K} , that is - there exists $b \in K^n$ such that $\mathcal{K} \models \phi_S(b) \wedge \nu(h(b)) < 0$.

The main theorem

Let P be the set of sums of square elements in L and let A be the O_K -Algebra generated by

$$I_{p_1, \dots, p_m} = \left\{ \frac{1}{1 + \sum_{i=1}^{2^m} s_i \prod_{j=1}^m p_j^{r_{i,j}}} \mid s_i \in P, r_{i,j} \in \{0, 1\} \right\} \cup \{f_1, \dots, f_{m'}\}$$

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Theorem

(N.Lavi and Y.Yaffe) For every $h \in L$ we have that h is OVF-Integral on $S_{\vec{p}, \vec{f}}$ if and only if $h \in \sqrt{\text{int} A_T}$

Let $I_{ord} = \left\{ \frac{1}{1+r} \mid r \in P \right\}$ and note that $I_{ord} \subseteq I_{p_1, \dots, p_m} \subseteq A$. We shall prove that I_{ord} indeed gives us the **extension** property. That is, That any $\tilde{\nu}$ extending ν such taht $A \subseteq O_{\tilde{\nu}}$ is an OVF-valuation.

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Lemma

Let $\langle K, \nu \rangle$ be an ordered valued field and let \leq_k be a linear order on the residue field k . Then there exists a linear order \leq_K which induces \leq_k and turning K into an ordered valued field.

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From the lemma above we learn that the existence of an order \leq_L extending \leq_K relies on the ability to extend the order to the residue field \tilde{k} .

Theorem

Let $\langle K, \nu, \leq_K \rangle$ be an ordered valued field and let $\tilde{\nu}$ an extension of ν to $L = K(x)$. Then $\tilde{\nu}$ is an OVF-valuation if and only if for every $r \in P$ we have that $\frac{1}{1+r} \in O_{\tilde{\nu}}$

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Proof.

For every order $<_L$ we have that $1 \leq 1 + r$ since for every sum of squares r we have that $r >_L 0$. Hence, since $\tilde{\nu}$ is an OVF-valuation we have that $\tilde{\nu}(\frac{1}{1+r}) \geq 0$.

For the other direction, suppose by contradiction that $\tilde{\nu}$ is not an OVF-valuation. Hence \tilde{k} is not formally real, so there exist $\alpha_1, \dots, \alpha_m \in \tilde{k}$ such that $\sum_{i=1}^m \alpha_i^2 = -1$. Choose $a_1, \dots, a_m \in O_{\tilde{\nu}}^\times$ such that $\text{res}(a_i) = \alpha_i$. Therefore $1 + \sum_{i=1}^m a_i^2 \in \mathcal{M}_L$ hence $\tilde{\nu}(\frac{1}{1 + \sum_{i=1}^m a_i^2}) < 0$. Setting $r = \sum_{i=1}^m a_i^2$ we get $\frac{1}{1+r} \notin O_{\tilde{\nu}}$. □

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Therefore, A has the extension property.