Positivity, Sums of Squares and Positivstellensätze for Noncommutative $*$-Algebras

Konrad Schmüdgen (Universität Leipzig)

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Dedicated to ALEXANDER PRESTEL on the occasion of his retirement

Contents

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Question:
How to generalize these concepts to noncommutative algebras?

Star Algebras

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Answer:

An algebra involution on $\mathcal{A}$ is needed!

An **involution** of the algebra $\mathcal{A}$ is a mapping $a \to a^*$ of $\mathcal{A}$ into $\mathcal{A}$ such that $(\lambda a + \mu b)^* = \bar{\lambda} a^* + \bar{\mu} b^*$, $(a^*)^* = a$ and $(ab)^* = b^* a^*$ for $a, b \in \mathcal{A}$ and $\lambda, \mu \in \mathbb{K}$.

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A $*$-**algebra** is an algebra equipped with an algebra involution.

In what follows we suppose that $\mathcal{A}$ is a unital $*$-algebra.
$\mathcal{A}_h = \{a \in \mathcal{A} : a = a^*\}$ is called the **hermitian part** of $\mathcal{A}$.

Classical Real Algebraic Geometry:

$\mathcal{A} = \mathbb{R}[x_1, \dots, x_d]$, $p^* := p$ or
$\mathcal{A} = \mathbb{C}[x_1, \dots, x_d]$, $p^* = \overline{p}$, where $\overline{p}(x) = \sum \overline{a_\alpha} x^\alpha$ for $p(x) = \sum a_\alpha x^\alpha$.

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Positivity of the Involution

All involutions occuring in this talk satisfy the following condition:

If $x_1^* x_1 + \cdots + x_k^* x_k = 0$ for $x_1, \dots, x_k \in \mathcal{A}$, then $x_1 = \cdots = x_k = 0$.

Quadratic Modules

Definition: Quadratic Modules

A **pre-quadratic module** of $\mathcal{A}$ is a subset $\mathcal{C}$ of $\mathcal{A}_h$ such that $\mathcal{C}+\mathcal{C}\subseteq\mathcal{C}$, $\mathbb{R}_+\cdot\mathcal{C}\subseteq\mathcal{C}$ and $b^*\mathcal{C}b\in\mathcal{C}$ for all $b\in\mathcal{A}$.

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Quadratic modules are important in theory of $*$-algebras where they have been called *m-admissible wedges*.

Each quadratic module gives an **ordering** $\preceq$ on the real vector space $\mathcal{A}_h$ by defining $a \preceq b$ (and likewise $b \succeq a$) if and only if $a - b \in \mathcal{C}$.

Algebraic Quadratic Modules

Definition: Pre-Quadratic Module $\mathcal{C}_{\mathcal{X}}$

If $\mathcal{X}$ is a subset of $\mathcal{A}_h$, then

$$\mathcal{C}_{\mathcal{X}} := \left\{ \sum_{j=1}^{s} \sum_{l=1}^{k} a_{jl}^* x_l a_{jl}; \quad a_{jl} \in \mathcal{A},\ x_l \in \mathcal{X},\ s, k \in \mathbb{N} \right\}$$

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All elements a^*a, where $a \in \mathcal{A}$, are called **squares** of $\mathcal{A}$.

The wedge

$$\sum \mathcal{A}^2 := \left\{ \sum_{j=1}^{n} a_j^* a_j; \quad a_1, \dots, a_n \in \mathcal{A}, n \in \mathbb{N} \right\}$$

of finite sums of squares is the **smallest quadratic module** of $\mathcal{A}$.

Quadratic Modules Defined by Representations

Let $\mathcal{D}$ be a $\mathbb{K}$-vector space with scalar product $\langle \cdot, \cdot \rangle$.

Definition: $*$-Representation

A $*$- **representation** of $\mathcal{A}$ on $\mathcal{D}$ is an algebra homomorphism π of $\mathcal{A}$ into the algebra of linear operators mapping $\mathcal{D}$ into itself such that $\pi(1)\varphi = \varphi$ and $\langle \pi(a)\varphi, \psi \rangle = \langle \varphi, \pi(a^*)\psi \rangle$ for all $\varphi, \psi \in \mathcal{D}$ and $a \in \mathcal{A}$.

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We write $\pi(a) \geq 0$ when $\langle \pi(a)\varphi, \varphi \rangle \geq 0$ for all $\varphi \in \mathcal{D}$.

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Definition: Quadratic Module $\mathcal{A}(\mathcal{S})_+$

For a family $\mathcal{S}$ of $*$-representations of $\mathcal{A}$, we define a **quadratic module**

$$\mathcal{A}(\mathcal{S})_+ := \{a \in \mathcal{A}_h : \pi(a) \geq 0 \ \text{ for all } \ \pi \in \mathcal{S}\}.$$

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If $b \in \mathcal{A}$ and $c \in \mathcal{A}(\mathcal{S})_+$, then $\langle \pi(b^*cb)\varphi, \varphi \rangle = \langle \pi(c)\pi(b)\varphi, \pi(b)\varphi \rangle \geq 0$ for $\pi \in \mathcal{S}$, so that $b^*cb \in \mathcal{A}(\mathcal{S})_+$.

Quadratic Modules Defined by States

A **state** of $\mathcal{A}$ is a linear functional f on $\mathcal{A}$ such that $f(1) = 1$ and $f(a^*a) \geq 0$ for all $a \in \mathcal{A}$.

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By the **GNS construction** each state arises in this manner:
For each state f there are a $*$-representation π_f and a unit vector $\varphi \in \mathcal{D}$ such that $\mathcal{D} = \pi_f(\mathcal{A})\varphi_f$ and

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Definition: Quadratic Module $\mathcal{A}(\mathcal{F})_+$

Let $\mathcal{F}$ be a set of states on $\mathcal{A}$ such that $f_a(\cdot) = f(a^*a)^{-1} f(a^* \cdot a)$ is in $\mathcal{F}$ for all $f \in \mathcal{F}$ and $a \in \mathcal{A}$ satisfying $f(a^*a) \neq 0$. Then

$$\mathcal{A}(\mathcal{F})_+ := \{a{=}a^* \in \mathcal{A} : f(a) \geq 0 \;\; \text{for} \;\; f \in \mathcal{F}\}$$

is a **quadratic module** of $\mathcal{A}$.

Positivstellensätze

There is an interplay between quadratic modules
which are defined in algebraic terms
(such as $\sum \mathcal{A}^2$ or $\mathcal{C}_{\mathcal{X}}$) and those
which are defined by means of $*$-representations or states
(such as $\mathcal{A}(\mathcal{S})_+$ or $\mathcal{A}(\mathcal{F})_+$) for some **distinguished** family of $*$-representations or states.

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This is one of the most interesting challenges for the theory!

Positivstellensätze show how elements of $\mathcal{A}(\mathcal{S})_+$ or $\mathcal{A}(\mathcal{F})_+$ can be representated by means of $\sum \mathcal{A}^2$ or $\mathcal{C}_{\mathcal{X}}$.

Maximal Quadratic Modules

Let $\mathcal{A}$ be a complex unital $*$-algebra.
A quadratic module $\mathcal{C}$ is called **proper** if $\mathcal{C} \neq \mathcal{A}_h$.
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If $\mathcal{C}$ is a maximal proper quadratic module of a commutative unital ring A, then $\mathcal{C} \cap (-\mathcal{C})$ is a prime ideal and $\mathcal{C} \cup (-\mathcal{C}) = A$.

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Theorem: Cimprič

Suppose $\mathcal{C}$ is a quadratic module of a complex $*$-algebra $\mathcal{A}$. Let $\mathcal{C}^0 := \mathcal{C} \cap (-\mathcal{C})$ and $\mathcal{I}_{\mathcal{C}} := \mathcal{C}^0 + i\mathcal{C}^0$.
(i) $\mathcal{I}_{\mathcal{C}}$ is a two-sided $*$-ideal of $\mathcal{A}$.
(ii) If $\mathcal{C}$ is a maximal proper quadratic module, $\mathcal{I}_{\mathcal{C}}$ is a prime ideal and

$$\mathcal{I}_{\mathcal{C}} = \{a \in \mathcal{A} : axx^*a^* \in \mathcal{C}^0 \text{ for all } x \in \mathcal{A}\}.$$

Role of the Family $\mathcal{S}$ of Representations

Lemma

Suppose $\mathcal{A}$ has a faithful $*$-representation and $\mathcal{A}$ is the union of finite dimensional subspaces E_n, $n \in \mathbb{N}$. Assume that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that the following is satisfied:

If $a \in \sum \mathcal{A}^2$ is in E_n, then we can write a as a finite sum $\sum_j \; a_j^* a_j$ such that all elements a_j are in E_{k_n}.

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In the commutative case this condition means that the quadratic module $\sum \mathcal{A}^2$ is *stable*.

Role of the Family $\mathcal{S}$ of Representations

Theorem: K.S. 1979

If $\mathcal{A}$ is the commutative $*$-algebra $\mathbb{C}[x_1, \dots, x_d]$, the Weyl algebra $\mathcal{W}(d)$, the enveloping algebra $\mathcal{E}(g)$ or the free $*$-algebra $\mathbb{C}\langle x_1, \dots, x_d \rangle$, then the cone $\sum \mathcal{A}^2$ is closed in the finest locally convex topology on $\mathcal{A}$.

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Corollary:

Let $\mathcal{A}$ be one of the $*$-algebras from the preceding theorem. For any $a \in \mathcal{A}_h$ the following are equivalent:

(i): $a \in \sum \mathcal{A}^2$.

(ii): $\pi(a) \geq 0$ for all (irreducible) $*$-representations π of $\mathcal{A}$.

(iii): $f(a) \geq 0$ for each (pure) state f of $\mathcal{A}$.

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In case of the free polynomial algebra $\mathcal{A} = \mathbb{C}\langle x_1, \dots, x_d\rangle$ the implication (ii)$\rightarrow$(i) is usally called *Helton's theorem.*

Some Examples

Example 1: Commutative Polynomial Algebra $\mathbb{R}[x_1, \dots, x_d]$

$\mathcal{S} := \{\pi_t : t \in \mathbb{R}\}$, where $\pi_t(p) = p(t), \mathcal{D} = \mathbb{C}$ or
$\mathcal{S} = \{\pi_\mu : \mu \in M(\mathbb{R}^d)\}$,
where $M(\mathbb{R}^d)$ is the set of positive Borel measure on $\mathbb{R}^d$ which have finite moments and $\pi_\mu(p)q = p{\cdot}q$ for $p, q \in \mathbb{R}[x_1, \dots, x_d] \subseteq L^2(\mathbb{R}^d, \mu)$.

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Example 2: Weyl Algebra
$\mathcal{W} = \mathbb{C}\langle a, a^* | aa^* - a^*a = 1\rangle = \mathbb{C}\langle p = p^*, q = q^* | pq - qp = -i\rangle$

$\mathcal{S} = \{\pi_0\}$, where π_0 is the **Bargmann-Fock representation** ($\pi_0(a)e_n = n^{1/2}e_{n-1}$, $\pi_0(a^*)e_n = (n+1)^{1/2}e_{n+1}$ on $l^2(\mathbb{N}_0)$) or the **Schrödinger representation** ($\pi_0(q)f{=}tf(t)$, $\pi_0(p)f{=}-if'(t)$ on $L^2(\mathbb{R})$).

Some Examples

Example 3: Enveloping Algebras $\mathcal{E}(g)$ of a Real Lie Algebra g with Involution $x^* = -x$ for $x \in g$

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Example 4: Free Polynomial Algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$ with Involution $x_j^* = x_j$

$\mathcal{S}$ is the set of all $*$-representations.

If $X_1, \dots, X_d$ are arbitrary bounded self-adjoint operators, then there is a $*$-representation π such that $\pi(x_1) = X_1, \dots, \pi(x_d) = X_d$.

What about Artin's Theorem in the Noncommutative Case?

Artin's Theorem on the solution of Hilbert's 17th problem:

For each nonnegative polynomial a on $\mathbb{R}^d$ there exists a nonzero polynomial $c \in \mathbb{R}[t]$ such that $c^2 a \in \sum \mathbb{R}[t]^2$.

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One might also think of

$$\sum_k c_k^* a c_k \in \sum \ \mathcal{A}^2,$$

but it can be shown that such a condition corresponds to a Nichtnegativstellensatz rather than a Positivstellensatz.

An Essential Difference

In the commutative case the relation $c^2 a \in \sum \mathbb{R}[x]^2$ implies that the polynomial a is nonnegative on $\mathbb{R}^d$.
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Example: Weyl Algebra

Let $\mathcal{A}$ be the Weyl algebra $\mathcal{W}$ and $\mathcal{S}=\{\pi_0\}$.
Set $N=a^*a$. Since $aa^*-a^*a=1$, we have $a(N-1)a^*=N^2+a^*a \in \sum \mathcal{A}^2$.
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Example: $*$-Algebra Generated by an Isometry

Let $\mathcal{A}$ be the $*$-algebra with a single generator a and relation $a^*a = 1$.
Then $p_0 := 1-aa^* \neq 0$ is a projection in $\mathcal{A}$ and $p_0 axa^* p_0 = 0 \in \sum \mathcal{A}^2$ for arbitrary $x \in \mathcal{A}$.
But elements of the form axa^* are in general not nonnegative in $*$-representations of $\mathcal{A}$.

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Problem:

Suppose that $c^*ac \in \sum \mathcal{A}^2$.

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If the representations are by bounded operators, then it suffices that

$$\text{ran } \pi(c^*) \subset \ker \pi(a).$$

Version 1 of Artin's Theorem in the Noncommutative Case

Version 1: Denominator Free

For any $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$ we have $a \in \sum \mathcal{A}^2$.

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Example 1: Fejer-Riesz Theorem:

Let $\mathcal{A} = \mathbb{C}\langle z, z^{-1} | z^* z = z^* z = 1 \rangle$ be the trigonometric polynomials.
Let $\mathcal{S} = \{\pi_w(p) = p(w, w^{-1}); w \in \mathbb{T}\}$ or $\mathcal{S} = \{\pi_0(z) = U\}$,
where $Ue_n = e_{n+1}$ on $l^2(\mathbb{Z})$ is the bilateratal shift.
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Example 2: Noncommutative Fejer-Riesz Theorem: Savchuk, K. S.

Let $\mathcal{A} = \mathbb{C}\langle s, s^* | ss^* = 1\rangle$ be the $*$-algebra generated by an isometry.
Let $\mathcal{S}$ be *all* $*$-representations of $\mathcal{A}$ or $\mathcal{S} = \{\pi_0\}$, where $\pi_0(s)e_n = e_{n+1}$ on $l^2(\mathbb{N}_0)$ is the unilateral shift.
If $\pi(p) \geq 0$ for all $\pi \in \mathcal{S}$, then there is $q \in \mathcal{A}$ such that $p = q^* q$.

Examples of Version 1

Example 3: Curves (C. Scheiderer)

Let $\mathcal{A} = \mathbb{R}[C]$ be the (real) coordinate algebra of an irreducible smooth affine curve C.
If C has at least one *nonreal* point at infinity, then version 1 holds.
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Example 4: Spherical Isometries (Helton/McCullough/Putinar)

$\mathcal{A} = \mathbb{C}\langle x_1, x_1^*, \dots, x_d, x_d^* | \;\; x_1^* x_1 + \cdots + x_d^* x_d = 1\rangle$.
Then version 1 holds.

Examples of Version 1

Example 5: Group Algebra of a Free Group

Let G be a free group and $\mathcal{A} = \mathbb{C}[G]$ be the group algebra with involution $g^* = g^{-1}$, $g \in G$. Then version 1 holds.

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Example 6: Matrices $M_n(\mathbb{C}[x_1])$

Djokovic (1976): Any element $A \in M_n(\mathbb{C}[x_1])_+$ is a square $A = B^*B$, $B \in M_n(\mathbb{C}[x_1])$. That is, version 1 holds.

Version 2 of Artin's Theorem in the Noncommutative Case

Version 2: With Denominators

For any $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$ there exists a $c \in \mathcal{A}$ such that c is not a zero divisor and $c^* a c \in \sum \mathcal{A}^2$.

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Example 1: Matrices of Poynomials
Gondard/Ribenboim (1974), Procesi/Schacher (1976)

$\mathcal{A} = M_n(\mathbb{R}[x_1, \ldots, x_d])$ and $\mathcal{S} = \{\pi_t((a_{ij})) = (a_{ij}(t));\, t \in \mathbb{R}^d\}$.
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Then version 2 holds.

Theorem: Savchuk, K.S.

Suppose $\mathcal{A}$ has no zero divisors and $\mathcal{A}\backslash\{0\}$ satifies a right Ore condition (e.g. for any $a \in \mathcal{A}$ and $s \in \mathcal{A}\backslash\{0\}$ there are $b \in \mathcal{A}$ and $t \in \mathcal{A}\backslash\{0\}$ such that $at = sb$). Let $\mathcal{A}$ be a $*$-algebra of operators on a pre-Hilbert space. Let $\mathcal{S}$ consists only of the identity representation.
If $\mathcal{A}$ satisfies version 2, then so $M_n(\mathcal{A})$.

Version 2 of Artin's Theorem in the Noncommutative Case

Example 2: Crossed Product Algebra

Let G be a finite group acting as $*$-automorphims $g \to \alpha_g$ of a unital $*$-algebra $\mathcal{A}$. The **crossed-product algebra** $\mathcal{A} \times_\alpha G$ is a unital $*$-algebra: As a vector space it is $\mathcal{A} \otimes \mathbb{C}[G]$, product and involution are given by

$$(a \otimes g)(b \otimes h) = a\alpha_g(b) \otimes gh, (a \otimes g)^* = \alpha_{g^{-1}}(a^*) \otimes g^{-1}, \ a, b \in \mathcal{A}, g, h \in G$$

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Suppose that $G = \mathbb{Z}_n$. If $\mathcal{A}$ satisfies version 2, then also $\mathcal{A} \times_\alpha \mathbb{Z}_n$.

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Idea of proof: Embedd $\mathcal{A} \times_\alpha \mathbb{Z}_n$ as a $*$-subalgebra of $M_n(\mathcal{A})$, construct a conditional expectation to $\mathcal{A} \times_\alpha \mathbb{Z}_n$ and apply the preceding theorem.
If σ is an $*$-automorphism of order 3, then $\mathcal{A} \times_\alpha \mathbb{Z}_3$ is the set of matrices

$$\begin{pmatrix} a & b & c \\ \sigma(c) & \sigma(a) & \sigma(b) \\ \sigma^2(b) & \sigma^2(c) & \sigma^2(a) \end{pmatrix}, \ a, b, c \in \mathcal{A}$$

Version 3 of Artin's Theorem in the Noncommutative Case

Denominators Sets and Preorderings

A **preorder** $\mathcal{C}$ is a quadratic module such that

$$c_1 c_2 \in \mathcal{C} \text{ for all } c_1, c_2 \in \mathcal{C},\ \ c_1 c_2 = c_2 c_1.$$

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Let $\mathcal{C}_{\mathcal{A}}$ be the smallest preorder on $\mathcal{A}$ and $a \in \mathcal{A}_h$. We form a **denominator set** $\mathcal{S}_a$:

(i) $a \in \mathcal{S}_a$. (ii) If $b \in \mathcal{S}_a$ and $x \in \mathcal{A}$, then $x^*bx \in \mathcal{S}_a$.
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Motivation: Suppose $\mathcal{A}$ is a $*$-algebra of *bounded* operators on a Hilbert space. If a is *positive*, then all elements of $\mathcal{S}_a$ and $\mathcal{C}_{\mathcal{A}}$ are positive as well.

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Version 3: Most General Denominators and Right Hand Sides

Suppose that $a = a^* \in \mathcal{A}$ such that $\pi(a) \geq 0$ for all $\pi \in \mathcal{S}$. Then there exist a $s_a \in \mathcal{S}_a$ such that $s_a \in \mathcal{C}_{\mathcal{A}}$.

Example: $x^*(c_1^*c_1 + c_2^*c_2)ax = y_2^*(c_3^*c_3(y_1^*(c_4^*c_4 + c_5^*c_5)y_1))y_2 + \cdots.$

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Examples of version 3: Talk by Y. Savchuk

An Example Concerning Versions 1, 2 and 3

Weyl Algebra $\mathcal{A} = \mathbb{C}\langle a, a^* | \; aa^* - a^*a = 1\rangle$

$\mathcal{S} = \{\pi_0\}$, where π_0 is the Bargmann-Fock representation $(\pi_0(a)e_n = n^{1/2}e_{n-1}, \, \pi_0(a^*)e_n = (n+1)^{1/2}e_{n+1}$ on $l^2(\mathbb{N}_0))$.

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(For instance, $(a^*a)(a^{*3}a^3)(N-1)(N-2) = (a^{*3}a^3)^2$.)

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Suppose that version 2 holds for $\mathcal{A}_1$ and $\mathcal{A}_2$. Does it hold for $\mathcal{A}_1 \otimes \mathcal{A}_2$?

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Does version 2 of Artin's theorem hold for the Weyl algebra (Example 2) or for Enveloping algebras (Example 3)?

For the Weyl algebra there are the following open problems:

Let $\mathcal{A} = \pi_0(\mathcal{W})$ be the $*$-algebra of differential operators

$$a = \sum_k p_k(t) \left(\frac{d}{dt} \right)^k$$

with polynomial coefficients acting on the Schwartz space $\mathcal{S}(\mathbb{R}) \subseteq L^2(\mathbb{R})$.

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Suppose $\langle a\varphi, \varphi\rangle \geq 0$ for $\varphi \in \mathcal{S}(\mathbb{R})$. Does there exist a nonzero element $c \in \mathcal{A}$ such that

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Problem 2: A Version of a Noncommutative Stengle Theorem

Suppose $\langle a\varphi, \varphi\rangle \geq 0$ for $\varphi \in C_0^\infty(0, +\infty)$. Does there exist an nonzero element $c \in \mathcal{A}$ such that

$$c^* ac = \sum a_i^* a_i + \sum b_j^* q b_j?$$

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In terms of the order relation $\geq$ defined by $\mathcal{C}$ condition (1) means that

$$\lambda \cdot 1 \leq a \leq \lambda \cdot 1.$$

This means that 1 is an order unit of the ordered vector space.

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$$\lambda \cdot 1 - a \in \mathcal{C} \quad \text{and} \quad \lambda \cdot 1 + a \in \mathcal{C}. \tag{1}$$

In terms of the order relation $\geq$ defined by $\mathcal{C}$ condition (1) means that

$$\lambda \cdot 1 \leq a \leq \lambda \cdot 1.$$

This means that 1 is an order unit of the ordered vector space.

Lemma (Vidav, K.S., Cimprič)

$\mathcal{C}$ is Archimedean iff (1) holds for a set of hermitian generators of $\mathcal{A}$.

Definition and Simple Properties

Lemma

$\mathcal{C}$ is Archimedean if and only if for element $a \in \mathcal{A}$ there exists a number $\lambda > 0$ such that $\lambda \cdot 1 - a^* a \in \mathcal{A}$.
(It suffices to know the latter condition for a set of generators.)

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$\mathcal{C}$ is *Archimedean* if and only if the unit 1 is an *internal point* of $\mathcal{C}$
(that is, for any $a \in \mathcal{A}_h$ there exists a number $\varepsilon_a > 0$ such that
$1 + \lambda a \in \mathcal{C}$ for all $\lambda \in \mathbb{R}$, $|\lambda| \leq \varepsilon_a$.)

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- **Eidelheit's separation theorem for convex sets applies!**

Abstract Positivstellensätze

$\mathcal{C}$-Positivity

A $*$-*representation* π is called $\mathcal{C}$-**positive** if $\pi(c) \geq 0$ for all $c \in \mathcal{C}$.
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Standard Example: Preorder of Basic Closed Semialgebraic Sets

Let $f{=}(f_1, \cdots, f_k)$ is a k-tuple of elements $f_j \in \mathcal{A} := \mathbb{R}[x_1, \ldots, x_d]$.

Basic closed semialgebraic set: $\mathcal{K}_f{=}\{t{\in}\mathbb{R}^d : f_1(t) \geq 0, \cdots, f_k(t) \geq 0\}$.

Preorder: $\mathcal{T}_f = \{ \sum_{\varepsilon_i \in \{0,1\}} \sum_{j=1}^n f_1^{\varepsilon_1} \cdots f_k^{\varepsilon_k} g_j^2 \ ; \ g_j \in \mathcal{A}, \ n \in \mathbb{N}\}$.

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Then we have :$\mathcal{T}_f$ **is Archimedean if and only if** $\mathcal{K}_f$ **is compact.**
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Assume that $\mathcal{C}$ is an Archimedean quadratic module of $\mathcal{A}$.

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For any element $a = a^* \in \mathcal{A}$ the following are equivalent:

(i) $a + \varepsilon{\cdot}1 \in \mathcal{C}$ for each $\varepsilon > 0$.

(ii) $\pi(a) \geq 0$ for each $\mathcal{C}$–positive $*$-representation π of $\mathcal{A}$.

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Condition (iii) means that $a \geq 0$ on $\mathcal{K}_f$ and (i) that $a + \varepsilon \in \mathcal{T}_f$ for any $\varepsilon > 0$. If we know that $\mathcal{T}_f$ is Archimedean (!), then the implication (iii)$\rightarrow$(i) is the assertion of the **Archimedean Positivstellensatz:** **If $\mathcal{K}_f$ is compact and $a \geq 0$ on $\mathcal{K}_f$, then $a + \varepsilon \in \mathcal{T}_f$.**

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Abstract Nichtnegativstellensatz: J. Cimprič 2005

For $a = a^* \in \mathcal{A}$ the following are equivalent:

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Proof of (i)$\rightarrow$(ii): Suppose that $\sum_k x_k^* a x_k = 1 + c$ with $c \in \mathcal{C}$. If π is a $\mathcal{C}$-positive $*$-representation and $\varphi \in \mathcal{D}(\pi)$, $\varphi \neq 0$, then

$$\sum_k \langle \pi(a)\pi(x_k)\varphi, \pi(x_k)\varphi \rangle = \sum_k \langle \pi(x_k^* a x_k)\varphi, \varphi \rangle$$
$$= \langle \pi(1 + c)\varphi, \varphi \rangle \geq \langle \pi(1)\varphi, \varphi \rangle = \langle \varphi, \varphi \rangle > 0.$$

Hence at least one summand $\langle \pi(a)\pi(x_k)\varphi, \pi(x_k)\varphi \rangle$ is positive.

$\mathcal{C}$-Positive Representations

Lemma

If $\mathcal{C}$ is an Archimedean quadratic module and π is a $\mathcal{C}$–positive $*$-representation of $\mathcal{A}$, then all operators $\pi(a)$, $a \in \mathcal{A}$, are bounded.

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Proof. Let $a \in \mathcal{A}$. Since $\mathcal{C}$ is Archimedean, there exists a $\lambda_a > 0$ such that $\lambda \cdot 1 - a^* a \in \mathcal{C}$. Therefore,

$$\langle(\pi(\lambda_a \cdot 1 - a^* a)\varphi, \varphi\rangle = \lambda_a ||\varphi||^2 - ||\pi(a)\varphi||^2 \geq 0$$

and hence $||\pi(a)\varphi|| \leq \lambda_a^{1/2} ||\varphi||$ for all $\varphi \in \mathcal{D}(\pi)$ and any $\mathcal{C}$-positive representation π.

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A $*$-algebra $\mathcal{A}$ is called **algebraically bounded** if the quadratic module $\sum \mathcal{A}^2$ is Archimedean.

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Since $*$-representations are always $\sum \mathcal{A}^2$-positive, each $*$-representation of an algebraically bounded $*$-algebra acts by bounded operators.

Examples of Archimedean Quadratic Modules

Example 1: Veronese Map

Let $\mathcal{A}$ be the complex $*$–algebra of rational functions generated by

$$x_{kl} := x_k x_l (1 + x_1^2 + \cdots + x_d^2)^{-1}\ ,\ \ k, l =, 1, \cdots , d,$$

where $x_0 := 1$. Since $1 = \sum_{r,s} x_{rs}^2 \geq x_{kl}^2 \geq 0$ for $k, l{=}1, \ldots, d$, the quadratic module $\sum \mathcal{A}^2$ is Archimedean and $\mathcal{A}$ is algebraically bounded.

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Proof: $1 - aa^* \in \sum \mathcal{A}^2$, $q^{-1} - a^*a \in \sum \mathcal{A}^2$.

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Example 3: Compact Quantum Group Algebras

The $*$–algebra $\mathcal{A}$ is span of elements v_{kl} satisfying the relation

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for all k . Hence $1 - v_{kl}^{*} v_{kl} \in \sum \mathcal{A}^2$, so $\mathcal{A}$ is algebraically bounded.

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For the quantum group $SU_q(2)$, $\mathcal{A}$ has two generators a and c satisfying

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Many **compact quantum spaces** have algebraically bounded coordinate $*$–algebras $\mathcal{A}$.

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Idea: Reduce SOS representations in $\mathcal{A}$ to SOS representations in $\mathcal{B}$.
Positivity on all characters of $\mathcal{B}$ requires positivity on all characters of $\mathcal{A}$ **and** positivity at χ_∞.

Basic Idea for Algebras of Fractions

Many algebras are not algebraically bounded, but they do have algebraically bounded fraction $*$-algebras with different classes of denominators!

Important Examples: **Weyl algebras** and **Enveloping algebras**

A Strict Positivstellensatz for the Weyl Algebra

Weyl algebra: $\mathcal{W} = \mathbb{C}\langle p, q | pq - qp = -i, p = p^*, q = q^* \rangle$

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Schrödinger representation π of $\mathcal{W}$:
$(\pi(q)f)(t) = tf(t)$ and $(\pi(p)f)(t) = -if'(t)$ on $\mathcal{S}(\mathbb{R}) = \mathcal{D}$ in $L^2(\mathbb{R})$.

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Then $\mathcal{A} := \pi(\mathcal{W})$ is the $*$-algebra of differential operators

$$a = \sum_{k=0}^{n} p_k(t)(\frac{d}{dt})^k$$

with polynomial coeffcient $p_k \in \mathbb{C}[t]$ acting on the Schwartz space $\mathcal{S}(\mathbb{R})$.

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Each element $c \in \mathcal{W}$, $c \neq 0$, can be written as

$$c = \sum_{j=0}^{d_1} \sum_{l=0}^{d_2} \gamma_{jl} p^j q^l = \sum_{n=0}^{d_2} f_n(p) q^n = \sum_{k=0}^{d_1} g_k(q) p^k,$$

where $\gamma_{jl} \in \mathbb{C}$, $f_n(p) \in \mathbb{C}[p]$, $g_k(q) \in \mathbb{C}[q]$ uniquely determined by c.
Set $d(c)=(d_1, d_2)$ if there are $j_0, l_0 \in \mathbb{N}_0$ such that $\gamma_{d_1, l_0} \neq 0$ and $\gamma_{j_0, d_2} \neq 0$.

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Theorem 1:

Let $c = c^*$ be a nonzero element of the Weyl algebra $\mathcal{A}$ with multi-degree $d(c) = (2n_1, 2n_2)$, where $n_1, n_2 \in \mathbb{N}_0$. Suppose that:

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Let $c = c^*$ be a nonzero element of the Weyl algebra $\mathcal{A}$ with multi-degree $d(c) = (2n_1, 2n_2)$, where $n_1, n_2 \in \mathbb{N}_0$. Suppose that:

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A Strict Positivstellensatz for the Enveloping Algebra of the $ax + b$-Group

Let $\mathcal{A}$ is the complex universal enveloping algebra of the Lie algebra of the affine group of the real line. Then $\mathcal{A}$ is the unital $*$-algebra with two generators $a = a^*$ and $b = b^*$ and defining relation

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Each nonzero element $c \in \mathcal{A}$ can be written as

$$c = \sum_{j=0}^{d_1} \sum_{l=0}^{d_2} \gamma_{jl} a^j b^l = \sum_{n=0}^{d_2} f_n(a) b^n = \sum_{k=0}^{d_1} g_k(b) a^n.$$

Here $\gamma_{jl} \in \mathbb{C}$ and $f_n(a)$, $g_k(b)$ are polynomials uniquely determined by c. Set $d(c){=}(d_1, d_2)$ if there are $j_0, l_0 \in \mathbb{N}_0$ such that $\gamma_{d_1,l_0}{\neq}0$ and $\gamma_{j_0,d_2}{\neq}0$.

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